# The Power of Random Symmetry-Breaking in Nakamoto Consensus

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#### Abstract

Nakamoto consensus underlies the security of many of the world's largest cryptocurrencies, such as Bitcoin and Ethereum. Common lore is that Nakamoto consensus only achieves consistency and liveness under a regime where the difficulty of its underlying mining puzzle is very high, negatively impacting overall throughput and latency. In this work, we study Nakamoto consensus under a wide range of puzzle difficulties, including very easy puzzles. We first analyze an adversary-free setting and show that, surprisingly, the common prefix of the blockchain grows quickly even with easy puzzles. In a setting with adversaries, we provide a small backwards-compatible change to Nakamoto consensus to achieve consistency and liveness with easy puzzles. Our insight relies on a careful choice of symmetry-breaking strategy, which was significantly underestimated in prior work. We introduce a new method—coalescing random walks—to analyzing the correctness of Nakamoto consensus under the uniformly-at-random symmetry-breaking strategy. This method is more powerful than existing analysis methods that focus on bounding the number of convergence opportunities.

## 1 Introduction

Nakamoto consensus [20], the elegant blockchain protocol that underpins many cryptocurrencies, achieves consensus in a setting where nodes can join and leave the system without getting permission from a centralized authority. Instead of depending on the identity of nodes, it achieves consensus by incorporating computational puzzles called proof-of-work [9] (also known as *mining*) and using a simple longest-chain protocol. Nodes in a network maintain a local copy of an append-only ledger and gossip messages to add to the ledger, collecting many into a block. A block consists of the set of records to add, a pointer to the previous block in the node's local copy of the ledger, and a nonce, which is evidence the node has done proof-of-work, or solved a computational puzzle of sufficient difficulty, dependent on the block. The node then broadcasts its local chain to the network. Honest nodes choose a chain they see with the most proof-of-work to continue building upon.

Previous work defined correctness and liveness in proof-of-work protocols (also referred to as the *Bitcoin backbone*) using three properties: *common-prefix*, *chain-quality*, and *chain-growth* [12, 15, 22]. Informally, common-prefix indicates that any two honest nodes share a common prefix of blocks, chain-growth is the rate at which the common prefix grows over time, and chain-quality represents the fraction of blocks created by honest nodes in a chain. In previous work, achieving these properties critically relied on the setting

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<sup>&</sup>lt;sup>1</sup>We use "longest chain" to mean the one with the most proof-of-work given difficulty adjustments, not necessarily the one with the most blocks, though without considering difficulty adjustments they are the same.

of the difficulty factor in the computational puzzles. We express this as p, the probability that any node will solve the puzzle in a given round. Previous work analyzing Nakamoto consensus has shown that for consistency and liveness p should be very small in relation to the expected network delay and the number of nodes [12, 22]. For example, mining difficulty in Bitcoin is set so that the network is only expected to find a puzzle solution roughly once every ten minutes.

Requiring a small p increases block time, removing a parameter for improving transaction throughput. One way to compensate is by increasing block size, which could result in burstier network traffic and longer transaction confirmation times for users. Newer chains which do not use proof-of-work seem to favor short block times, probably because users value a fast first block confirmation: in EOS, blocks are proposed every 500 milliseconds [10] and Algorand aims to achieve block finality in 2.5 seconds [19], whereas in Bitcoin blocks only come out every ten minutes.

Common belief is that larger p fundamentally constrains chain growth (i.e., the growth of the common prefix), even in the absence of an adversary, due to the potential of increased forking: nodes will find puzzle solutions (and thus blocks) at the same time; because of the delay in hearing about other nodes' chains nodes will build on different chains, delaying agreement. Another common conjecture, explicitly mentioned in [12], is that the choice of  $symmetry-breaking\ strategies$ , or ways honest nodes choose among multiple longest chains, is not relevant to correctness.

In this paper, we show that these common beliefs are incorrect. In particular, we show that when p is beyond the well-studied region even the simple strategy of choosing among chains of equal length randomly fosters chain growth, especially in the absence of adversaries.

Contributions. In this work, we formally analyze Nakamoto consensus under a wide range of p including large p. We confirm previous (informal) analysis that Nakamoto consensus requires small p in the presence of adversaries, but show that surprisingly, it does not in a setting without adversaries, even if p=1 (all nodes mine blocks every round) with a minor change in nodes' symmetry-breaking strategy. Previous work assumed the requirement of convergence opportunities, a period when only one honest node mines a block, in order to achieve consistency [18, 22]; we show that in fact convergence opportunities are not required for common-prefix and chain growth. With an additional backwards-compatible modification to Nakamoto consensus, we can derive a bound on the chain growth for a wider range of p (including large p) in a setting with adversaries. Our key idea in this modification is to introduce a verifiable delay function [5] to prevent the adversaries from extending a chain by multiple blocks in a round. Our analysis is based on a new application of a well-known technique, coalescing random walks. To our knowledge this is the first application of coalescing random walks to analyze the common-prefix and chain quality of Bitcoin and other proof-of-work protocols. We thoroughly analyze Nakamoto consensus with the uniformly-at-random symmetry-breaking strategy and discuss different symmetry-breaking strategies including first-seen, lexicographically-first, and global-random-coin.

In summary, our contributions are as follows:

- A new approach for analyzing the confirmation time of the Bitcoin protocol under the uniformly-atrandom symmetry-breaking strategy in the adversarial-free setting via *coalescing random walks*. Our analysis works for a new region of p, and shows that previous works' requirement for *convergence* opportunities was unneeded.
- New notions of *adversarial advantages* and *coalescing opportunities* to provide a more general analysis of common-prefix and chain growth in Nakamoto consensus in the presence of adversaries.

Related Work. Proofs-of-work were first put forth by Dwork and Naor [9]. Garay, Kiayias, and Leonardas [12] provided the first thorough analysis of Nakamoto's protocol in a synchronous static setting, introducing the ideas of common-prefix, chain quality and chain growth. Later work [15] extended the analysis to a variable difficulty function. Pass, Seeman, and shelat [22] extended the idea of common-prefix to future self-consistency, and provided an analysis of Nakamoto consensus in the semi-synchronous setting with an adaptive adversary. Several additional papers used this notion of future self-consistency [18, 30]. [18, 22] relied on convergence opportunities, or rounds where only one node mines a block, to analyze chain growth. In this work we show that convergence opportunities are not required for chain growth, and relying on them underestimates chain growth with high p; in the adversary-free setting we show chain growth even with p = 1

(no convergence opportunities; all nodes mine a block every round). Other work considered the tradeoffs between chain growth and chain quality [15,17,22,24,29]; however, to the best of our knowledge, none of these works considered different symmetry breaking strategies to enable faster chain growth while maintaining chain quality. In our paper, we thoroughly explore this domain. Another line of work [11,27] considers how the uniformly-at-random symmetry breaking strategy affects incentive-compatible selfish mining attacks; our analysis applies to general attacks.

Random walks have been used to analyze the probability of consistency violations in proofs-of-stake protocols [3]; ours is the first work that uses coalescing random walks to analyze the common-prefix and chain quality of Bitcoin and other proof-of-work protocols.

# 2 Model and Definitions

In this section, we present the specific model we use and briefly describe the Bitcoin cryptosystem. We follow the formalization presented in [15, 18, 22].

Network and Computation Model. Following previous work [12, 14, 15, 22, 26, 30], we consider a synchronous network where nodes send messages in synchronous rounds, i.e.,  $\Delta = 1$ ; equivalently, there is a global clock and the time is slotted into equal duration rounds. Each node has identical computing power. Notably, the synchronous rounds assumption is significantly more relaxed than assuming  $\Delta = 0.2$  Our model operates in the *permissionless setting*. This means that any miner can join (or leave) the protocol execution without getting permission from a centralized or distributed authority. For ease of exposition, we assume the number of participants remains n. Our results can be easily generalized to handle perturbation in the population size by a stochastic dominance argument as long as the population size does not deviate too far from n, and the proportion of Byzantine participants does not increase due to the perturbation.

Adversary Model. Throughout this paper, we assume that all Byzantine nodes are controlled by a probabilistic polynomial time (PPT) adversary  $\mathcal{A}$  that can coordinate the behavior of all such nodes.  $\mathcal{A}$  operates in PPT which means they have access to random coins but can only use polynomial time to perform computations. At any time during the run of the protocol,  $\mathcal{A}$  can corrupt up to b nodes at any point in time where b is a parameter that is an input to the protocol. The corrupted nodes remain corrupted for the remainder of the protocol. Finally,  $\mathcal{A}$  cannot modify or delete the messages sent by honest nodes, but can read all messages sent over the network and arbitrarily order the messages received by any honest nodes.

#### 2.1 Bitcoin Cryptosystem

A blockchain protocol is a stateful algorithm wherein each node maintains a local version of the blockchain  $\mathcal{C}$ . Each honest node runs its own homogeneous version of the blockchain protocol. Nodes receive messages from the environment  $\mathcal{Z}(1^{\lambda})$ , where  $\lambda$  is the security parameter chosen based on the population size n. The environment is responsible for all the external factors related to a protocol's execution. For example, it provides the value of b to the nodes. Detailed description of the environment can be found in [22].

The protocol begins by having the environment  $\mathcal{Z}$  initialize n nodes. The protocol proceeds in synchronous rounds; at each round r, each node receives a message from  $\mathcal{Z}$ . In each round, an honest node attempts to mine a block containing its message to add to its local chain. We provide formal definitions of the Bitcoin cryptosystem below.

#### **Blocks and Blockchains**

A blockchain  $C \triangleq B_0B_1B_2\cdots B_\ell$  for some  $\ell \in \mathbb{N}$  is a chain of blocks. Here  $B_0$  is a predetermined *genesis block* that all chains must build from. A *block*  $B_\ell$ , for  $\ell \geq 1$ , is a triple  $B_\ell = \langle s, x, \mathsf{nce} \rangle$ , where s, x,  $\mathsf{nce} \in \{0, 1\}^*$  are three binary strings of arbitrary length. Specifically, s is used to indicate this block's predecessor, s is the text of the block containing the message (e.g. transactions) and other metadata, and  $\mathsf{nce}$  is a *nonce* chosen by a node.

#### Proofs-of-Work

The Bitcoin cryptosystem crucially uses nonces as proofs-of-work for determining whether a block can be

<sup>&</sup>lt;sup>2</sup>In fact, the analysis based on Poisson race [2,21] essentially assumes all mined blocks can be ordered in a globally consistent way, i.e.,  $\Delta = 0$ , which does not hold in our synchronous network model.

legally added to a chain.<sup>3</sup> Proof-of-work (PoW) is rigorously defined in previous work [12, 14, 15, 22, 26, 30] based on the use of the *random oracle model*.

**Definition 1** (Random Oracle Model). A random oracle  $\mathcal{H}: \{0,1\}^* \to \{0,1\}^{\lambda}$  on input  $x \in \{0,1\}^*$  outputs a value selected uniformly at random from  $\{0,1\}^{\lambda}$  if x has never been queried before. Otherwise, it returns the previous value returned when x was queried last.

**Definition 2** (Bitcoin PoW). All nodes access a common random oracle  $\mathcal{H}: \{0,1\}^* \to \{0,1\}^{\lambda}$ . We say a node successfully performs a PoW with *proof*  $x \in \{0,1\}^*$  if  $\mathcal{H}(x) \leq D$ .

**Definition 3** (Valid Chain). A blockchain  $C = B_0 B_1 \cdots B_\ell = B_0 \langle s_1, x_1, \mathsf{nce}_1 \rangle \cdots \langle s_\ell, x_\ell, \mathsf{nce}_\ell \rangle$  is valid with respect to a given puzzle difficulty level  $D \in \{1, \dots, 2^{\lambda}\}$  if the following hold: (1)  $\mathcal{H}(B_0) = s_1$  and  $\mathcal{H}(B_{\ell'}) = s_{\ell'+1}$  for  $\ell' = 1, \dots, \ell-1$ ; and (2)  $\mathcal{H}(B_{\ell'}) \leq D$  for  $\ell' = 0, \dots, \ell$ .

#### Longest Chain Rule

The length of a valid chain C is the number of blocks it contains. We refer to the local version of the blockchain kept by node i as the local chain at node i, denoted by  $C_i$ . In each round r, node i tries to mine a block via solving a PoW puzzle with the specified difficulty D. If a block is successfully mined, then node i extends its local chain with this block and broadcasts its updated local chain to all other nodes in the network, which will be delivered at each node at the beginning of the next round. At the beginning of the next round, before working on PoW, node i updates its local chain to be the longest chain it has seen. If there are many longest chains, node i chooses one of them uniformly at random.

For ease of exposition, henceforth,  $C_i$  is referred to the local chain at the end of a round;  $C_i(t)$  is the local chain of node i at the end of round t. Equivalent to using the difficulty parameter D, one can instead consider  $p \triangleq D/2^{\lambda}$ . The notion of p used in lieu of D has been considered in [12,14,15,18,22,26] to simplify notation. Henceforth, we will quantify the algorithm performance in terms of p rather than D and  $\lambda$ .

We use the phrase with overwhelming probability throughout this paper. With overwhelming probability is defined as with probability at least  $1 - \frac{1}{\text{poly}(\lambda)^c}$  for any constant  $c \geq 1$ . We use the phrase with all but negligible probability in  $\lambda$  to mean that the probability is upper bounded by some negligible function  $\nu(\lambda)$  on  $\lambda$  (defined in Definition 4).

**Definition 4** (Negligible Probability). A function  $\nu$  is negligible if for every polynomial  $p(\cdot)$ , there exists an N such that for all integers n > N, it holds that  $\nu(n) < \frac{1}{p(n)}$ . We denote such a function by negl. An event that occurs with negligible probability occurs with probability negl(n).

#### 2.1.1 Properties of the Protocol

In this paper, we will analyze the Nakamoto consensus in terms of two characteristics (generalized from definitions in [12,18,30]). The common prefix is defined as a sub-chain that is a common prefix of the local chains of all honest nodes at the end of a round. The two properties maximal common prefix and maximal inconsistency are defined intuitively as: the maximal prefix that is the same across all honest chains and the maximal number of blocks in any honest chain that is not shared by all other honest chains, respectively.

**Property 5** (Maximal common-prefix and maximal inconsistency). Given a collection of chains  $\mathcal{C} = \{\tilde{C}_1, \dots, \tilde{C}_m\}$  that are kept by honest nodes, the maximal common-prefix of chain set  $\mathcal{C}$ , denoted by  $P_{\mathcal{C}}$ , is defined as the longest common-prefix of chains  $\tilde{C}_1, \dots, \tilde{C}_m$ . The maximal inconsistency of  $\mathcal{C}$ , denoted by  $I_{\mathcal{C}}$ , is defined as

$$\max_{i:1 \le i \le m} \left| \tilde{C}_i - P_{\mathcal{C}} \right|,\tag{1}$$

where  $\tilde{C}_i - P_{\mathcal{C}}$  is the sub-chain of  $\tilde{C}_i$  after removing the prefix  $P_{\mathcal{C}}$  and  $|\cdot|$  denotes the length of the chain, i.e., the number of blocks in the chain.

<sup>&</sup>lt;sup>3</sup>Note that in practice, the nonce is effectively concatenated with a miner's public key (included in the *coinbase* transaction) to ensure unique queries. The public key does *not* need to be verified. Importantly, this means that the miner can just generate a (pk, sk) pair on their local computer without the need to verify that identity with a third-party authority.

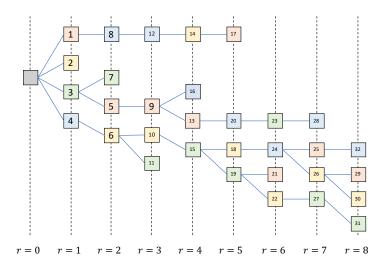


Figure 1: Example growth of a set of chains starting with the genesis block at round r = 0. Here, in this example p = 1, n = 4, and b = 0.

# 3 Fundamental Limitations of Existing Approaches

To the best of our knowledge, existing work assumes extremely small p. In fact, the seemingly mild honest majority assumption in [13,23] also implicitly assumes small p.

**Proposition 6.** If the honest majority assumption in [13] holds, then  $p \leq \frac{n-2b}{2(n-b)^2}$ .

A formal statement of the honest majority assumption and the proof of Proposition 6 can be found in Appendix B. Note that the upper bound in this proposition is only a necessary condition. Having p satisfy this condition does not guarantee protocol correctness.

**Remark 7.** Proposition 6 implies that in the vanilla Nakamoto consensus protocol, unless  $\frac{b}{n}$  is non-trivially bounded above from  $\frac{1}{2}$ , p needs to be extremely low – even much lower than the commonly believed  $\Theta(\frac{1}{n})$ . See Appendix B for detailed arguments.

To the best of our knowledge, most of the existing analyses focus on bounding the number of "convergence" opportunities", which for  $\Delta = 1$  is defined as the number of rounds in which exactly one honest node mines a block, and for general  $\Delta$ , it is defined as the global block mining pattern that consists of (i) a period of  $\Delta$  rounds where no honest node mines a block, (ii) followed by a round where a single honest player mines a block, (iii) and, finally, another  $\Delta$  rounds of silence from the honest nodes [18, 22]. Obviously, guaranteeing sufficiently many convergence opportunities necessarily requires p to be small; in the extreme case when p=1 there will be no convergence opportunities at all. An important insight from our results is that convergence opportunities are not necessary for common-prefix growth. This is illustrated Fig. 1 which depicts the chain growth when there are 4 honest nodes and p=1. Each node mines a block every round and each is associated with a color. In particular, blocks 1, 5, 9, 13, 17, 21, 25, 29 are mined by the pink node, blocks 4, 8, 12, 16, 20, 24, 28, 32 are mined by the blue node, etc. In each round, each node chooses one of the existing longest chains uniformly at random to extend. As shown in Fig. 1, there are no convergence opportunities in any of these 8 rounds and the four nodes never choose the same chain to extend. However, instead of the trivial common prefix (the genesis block) the longest chains at the end of round 8 (the four chains ending with blocks 32, 29, 30, and 31, respectively) share the common prefix genesis  $\rightarrow 4 \rightarrow 6 \rightarrow 10 \rightarrow 15$ . In general, as we show in Section 4, even for the extreme case when p=1, the common prefix of the longest chains still grows as time goes by.

# 4 Uniformly-at-Random Symmetry-Breaking Strategy

Bitcoin uses the *first-seen* symmetry-breaking strategy; nodes will only switch to a new chain with more proof-of-work than their current longest chain. In this section, we investigate the power of the uniformly-at-random symmetry-breaking strategy, in which each honest node chooses one of its received longest chains uniformly at random to extend upon – independently of other nodes and independently across rounds. We choose to start with the uniformly-at-random strategy because (1) it is easy to implement, especially in a distributed fashion, and (2) despite its simplicity, it is very powerful in fostering chain growth.

For ease of exposition, we first present our results in the adversary-free setting (Sections 4.1 and 4.2) and then in the adversary-prone setting (Section 4.3).

## 4.1 Warmup: p = 1 and Adversary-Free

Even the adversary-free setting (i.e., b = 0) is surprisingly non-trivial to analyze. Hence we build insights by first considering the simpler setting where p = 1 as a warmup.

**Theorem 8.** Suppose that p = 1 and b = 0. Then for any given round index  $t \ge 1$ , in expectation, the local chains at the honest nodes share a common prefix of length t + 1 - O(n).

**Remark 9.** In Theorem 8, the expectation is taken w. r. t. the randomness in the symmetry breaking strategy. Theorem 8 says that large p indeed boosts the growth of the common prefix among the local chains kept by the honest nodes, and that, though temporal forking exists among local chains kept by the honest nodes, such forking can be quickly resolved by repetitive symmetry-breaking across rounds.

The following definition and theorem are useful to see the intuitions of Theorem 8.

**Definition 10** (Coalescing Random Walks  $[1]^4$ ). In a coalescing random walk, a set of particles make independent random walks on a undirected graph G = (V, E) with self-loops. Whenever one or more particles meet at a vertex, they unite to form a single particle, which then continues the random walk through the graph. We define the *coalescence time*, denoted by  $C_G$ , to be the number of steps required before all particles merge into one particle.

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Theorem 11 ([1][7]). If G = (V, E) is complete, then \mathbb{E}[C_G] = O(n).
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 $(\mathbf{yellow}, 2) \rightarrow (\mathbf{blue}, 1) \rightarrow (\mathbf{gray}, 0).$ 

In the proof of Theorem 8, we build up the connection between the longest chains and the backwards coalescing random walks on complete graphs, and show that the maximal inconsistency among n longest chains turns out to be the same as the number of steps it takes n random walks on the n-complete graph to coalesce into one. Finally, we use the existing results on coalescing random walks to conclude.

Main proof ideas of Theorem 8. We cast our proof insights via an example presented in Fig. 1. In this figure, there are four miners. For ease of exposition, we use the colors pink, yellow, green, and blue to represent each of the miners, respectively. As shown in Fig. 1, there are 4 longest chains at the end of round 8 and these chains share a maximal common prefix ending at block 15. The maximal inconsistency of these 4 longest chains is 4; that is, these 4 longest chains are NOT inconsistent with each other until the most recent 4 blocks of each chain. For expository convenience below, instead of using numbers to represent each of the blocks, we use the tuple (color, r) to represent a block that is mined by a certain miner at round r. The maximal inconsistency of the longest chains can be characterized by the coalescing time on complete graphs. To see this, let's consider the four longest chains held by honest miners during round 8 backwards. Backwards-Chain #1:  $(blue, 8) \rightarrow (pink, 7) \rightarrow (blue, 6) \rightarrow (yellow, 5) \rightarrow (green, 4) \rightarrow (yellow, 3) \rightarrow (yellow, 2) \rightarrow (blue, 1) \rightarrow (gray, 0)$ , which can be read as "block (blue, 8) is attached to block (pink, 7) which is further attached to block (blue, 6) ... attached to the genesis block (gray, 0)."

Backwards-Chain #2:  $(pink, 8) \rightarrow (yellow, 7) \rightarrow (pink, 6) \rightarrow (green, 5) \rightarrow (green, 4) \rightarrow (yellow, 3) \rightarrow (yellow, 2) \rightarrow (blue, 1) \rightarrow (gray, 0)$ .

Backwards-Chain #3:  $(yellow, 8) \rightarrow (yellow, 7) \rightarrow (pink, 6) \rightarrow (green, 5) \rightarrow (green, 4) \rightarrow (yellow, 3) \rightarrow (yellow, 2) \rightarrow (blue, 3)$ .

<sup>&</sup>lt;sup>4</sup>The original definition given in [1] assumes no self-loops, but its analysis applies to the graphs with self-loops.

Backwards-Chain #4:  $(green, 8) \rightarrow (green, 7) \rightarrow (yellow, 6) \rightarrow (green, 5) \rightarrow (green, 4) \rightarrow (yellow, 3) \rightarrow (yellow, 2) \rightarrow (blue, 1) \rightarrow (gray, 0)$ .

Since p=1 and there is no adversary, the number of longest chains received by each honest node at each round is n. Under our symmetry-breaking rule, in each round t, each miner chooses which of the longest chains received at the beginning of round t to extend on uniformly-at-random. Thus, neither the previous history up to round t nor the future block attachment choices after round t affects the choice of the chain extension in round t. Reasoning heuristically<sup>5</sup>, we can view each of the backwards-chain as a random walk on a 4-complete graph with vertex set {pink, yellow, green, blue}. In particular, Backwards-Chain #1 can be viewed as a sample path of a random walk starting at the blue vertex, then moves to the pink vertex, then back to the blue vertex etc., and finally to the blue vertex. Similarly, Backwards-Chains #2, #3, and #4 can be viewed as the sample paths of three random walks starting at the pink vertex, yellow vertex, and green vertex, respectively. These four random walks (starting at four different vertices) are not completely independent. For any pair of random walks, before they meet, they move on the graph independently of each other; whenever they meet, they move together henceforth. Concretely, backwards-chains 2 and 3 meet at (yellow, 7) and these chains are identical starting from block (yellow, 7); this holds similarly for other pairs of backwards chains. Finally, these four backward chains all meet at the block (green, 4) and move together henceforth. Notably, this block is exactly the last block in the maximal common prefix of the four longest chains of round 8. Thus, the maximal inconsistency among the longest chains of round 8 is identical to the number of backwards steps it takes for all these four random walks to coalesce into one. This relation is not a coincidence. It can be shown (detailed in the proof of Theorem 8) that this identity holds for general n. Formal proof of Theorem 8 can be found in Appendix C.

#### 4.2 General p: Adversary-Free

The analysis for general p is significantly more challenging than that of p = 1 in two ways: (1) we need to repeatedly apply coupling arguments; and (2) we need to characterize the coalescence time of a new notion of coalescing random walks (the lazy coalescing random walks), the latter of which could be of independent interest for a broader audience.

**Theorem 12.** Suppose that  $np = \Omega(1)$ . If  $p < \frac{4 \ln 2}{n}$ , in expectation, at the end of round t, the local chains at the nodes share a common prefix of length  $(1 + (1 - (1 - p)^n)t) - O(\frac{1}{npe^{-np}})$ . If  $p \ge \frac{4 \ln 2}{n}$ , in expectation, at the end of round t, the local chains at the nodes share a common prefix of length  $(1 + (1 - (1 - p)^n)t) - O(\frac{2np}{(1-2\exp(-\frac{1}{3}np))})$ .

**Remark 13.** The expression of the common prefix length in Theorem 12 contains two terms with the first term (i.e.,  $(1 + (1 - (1 - p)^n)t)$ ) being the only term that involves t. Intuitively, from this term, we can read out the common prefix length growth rate w.r.t. t. The second term (which is expression in terms of Big-O notation) can be interpreted as a quantification of the maximal inconsistency of the honest chains.

Now we further interpret these two terms via simplifying the expression using the inequalities  $(1 - np) \le (1 - p)^n \le \exp(-np)$ .

- (1) When np = o(1), it is true that  $(1-p)^n \approx (1-np)$  for large n, which implies that  $(1-(1-p)^n)t \approx npt = o(t)$ , i.e., the common prefix grows at a speed o(t). The maximal inconsistency bound  $O(\frac{1}{npe^{-np}})$  is not tight. Nevertheless, via a straightforward calculation, we know that the maximal inconsistency is O(1).
- (3) When  $np = \omega(1)$ , we have  $0 \le (1-p)^n \le \exp(-np) \to 0$  as  $np \to \infty$ . Thus the common-prefix grows at the speed  $(1-(1-p)^n)$   $t \approx t = \Omega(t)$  with maximal inconsistency O(np) for sufficiently large np.
- (4) When  $np = c \in (0,1)$ , it is true that  $(1-p)^n = (1-c/n)^n \to \exp(-c)$  as  $n \to \infty$ . The common-prefix grows at the speed of  $\Theta(t)$  for sufficiently large n and the maximal inconsistency is O(1).

Overall, when np gets larger, the common-prefix growth increases and the maximal inconsistency grows at a much slower rate.

The following definition and lemma are used in proving Theorem 12. This lemma could be of independent interest to a broader audience and its proof can be found in the appendix.

<sup>&</sup>lt;sup>5</sup>Formally shown in the proof of Theorem 8 via introducing an auxiliary process.

**Definition 14** (Lazy coalescing random walk). For any fixed  $u \in (0,1)$ , we say n particles are u-lazy coalescing random walks if for each step: with probability (1-u), each particle stays at its current location; with probability u, each particle moves to an adjacent vertex picked uniformly at random. If two or more particles meet at a location, they unite into a single particle and continue the procedure. The *coalescence time* is the same as that in Definition 10.

**Lemma 15.** Suppose that G is a complete graph of size  $|V| = n_g$  (where  $n_g \ge 2$ ) with self-loops. For any  $u \in (0,1)$ , the coalescence time of the u-lazy coalescing random walks is  $C_G(n_g) = O(n_g/u)$ .

Proof Sketch of Theorem 12. When  $p < \frac{4 \ln 2}{n}$ , we can use Poisson approximation to approximate the distribution of number of blocks in each round. A straightforward calculation shows that the probability of having exactly one block in a round is  $np \exp(-np)$ . Thus, in expectation, the maximal inconsistency is  $O\left(\frac{1}{np \exp(-np)}\right)$ . Henceforth, we restrict our attention to the setting where  $p \geq \frac{4 \ln 2}{n}$  and quantify the expected maximal inconsistency among the longest chains of round t. It is attempting to apply arguments similar to that in the proof of Theorem 8 and derive a bound on the maximal inconsistency via stochastic dominance. However, the obtained bound on the maximal inconsistency is O(n) which could be extremely loose for a wide range of p. Nevertheless, based on the insights obtained in this coarse analysis, we can come up with a much finer-grained analysis and obtain the bound in Theorem 12. Similar to the proof of the special case when p=1, in our fine-grained analysis for general  $p \in (0,1)$ , we couple the growth of the common prefix in Nakamoto protocols with the coalescing time random walks on complete graphs. The major differences from the proof of p=1 are: (1) instead of the standard coalescing random walks, we need to work with a lazy version of it, formally defined in Definition 14; (2) there is no fixed correspondence between a color and a node – in our proof of general p, the correspondence is round-specific rather than fixed throughout the entire dynamics; (3) there is no bijection between a sample path of the Nakamoto dynamics and that of the backwards coalescing random walks, thus, we need to rely on stochastic dominance to build up the connection of these two dynamics.

## 4.3 General p: Adversary-Prone

Throughout this section, we assume p < 1. In this subsection, we consider adversary-prone systems, i.e., b > 0. Simple concentration arguments show that when  $bp \ge (1 + 2c)$  for any given  $c \in (0,1)$ , using vanilla Nakamoto consensus the chain quality could be near zero. To make larger p feasible, we introduce a new assumption—Assumption 16—which we then remove in Section 5 by providing a construction that ensures Assumption 16 with all but negligible probability. Specifically, we use a cryptographic tool called a VDF to ensure that over a sufficiently long time window, the corrupt nodes can only collectively extend a chain by more than one block in a round with negligible probability.

**Assumption 16.** In each round, a chain can be extended by at most 1 block.

To strengthen the protocol robustness, we make the additional minor modification requiring each honest node to selectively relay chains at the *beginning* of a round.

Selective relay rule: At each honest node i, for each iteration  $t \geq 1$ : Node i looks at the chains it received in the previous round t-1, and if any of them are longer than its own local longest chain, it not only chooses one of the longest chains to replace its local one, it also broadcasts it to other nodes before it begins mining in round t.

As implied by our proof, this modification can reduce the maximal difference between the lengths of the longest chains kept by the honest nodes and by the corrupt nodes. Intuitively, if the adversary sends two chains of different lengths to two different groups of honest nodes, with the selective relay rule, only the longer chain would survive in this round. Notably, it is possible that none of them survive in this round. Even with the assurance guaranteed by Assumption 16, compared with the adversary-free settings, the analysis for the adversary-prone setting is challenging. This is because the corrupt nodes could deviate from the specified symmetry breaking rule. For example, a corrupt node can choose not to extend its longest chain, or can choose from its set of longest chains in any way that provides advantage. In addition, a corrupt node can hide blocks it has mined from the honest nodes for as long as it wants, or from some subset of the honest nodes during a round.

For simplicity and for technical convenience, we assume that a corrupt node randomly chooses among longest chains that end with an honest block. This assumption is only imposed in the rare event when simultaneously both the adversary has no adversary advantage (see Definition 17) and only honest nodes mine blocks in the most recent nonempty round.

In contrast to the adversary-free setting where the lengths of honest nodes' local chains differ by at most 1, in the presence of an adversary, such difference could be large. To precisely bound this difference, we introduce a random process we call *adversary advantage*:

**Definition 17** (Adversary advantage). Let  $\{\mathcal{N}(t)\}_{t=0}^{\infty}$  be the random process defined as

- $\mathcal{N}(0) = 0$ , and
- for  $t \geq 1$ ,

$$\mathcal{N}(t) = \begin{cases} \mathcal{N}(t-1) + 1, & \text{if only corrupt nodes found blocks in round } t; \\ \max\{\mathcal{N}(t-1) - 1, \ 0\}, & \text{if only honest nodes found blocks in round } t; \\ \mathcal{N}(t-1), & \text{otherwise.} \end{cases}$$

Note that the random process  $\{\mathcal{N}(t)\}_{t=0}^{\infty}$  is independent of the adversarial behaviors of the corrupt nodes. To make the discussion concrete, we introduce the following definition.

**Definition 18.** The length of the longest chains kept by the honest nodes at round t is defined as the length of the longest local chains kept by honest nodes at the end of round t.

**Lemma 19.** For any  $t \ge 1$ , at the end of round t, the length of the longest chains kept by the adversary – henceforth referred to as an adversarial longest chain of round t – is at most  $\mathcal{N}(t)$  longer than the length of a chain kept by an honest node.

Proof of Lemma 19 can be found in Appendix E. From its proof, we can deduce an attacking strategy of the adversary that meets the upper bound in Lemma 19. The following lazy random walk, referred to as coalescing opportunities, is important in our analysis. It can also be used to quantify the chain quality.

**Definition 20.** Let  $t_1, t_2, \cdots$  be the rounds in which at least one node mines a block with the understanding that  $t_0 = 0$ . Let  $\mathcal{J}(m)$  be a random walk defined as

$$\mathcal{J}(m) = \begin{cases} 0, & \text{if } m = 0; \\ \mathcal{J}(m-1) + 1, & \text{if only honest nodes mine a block during round } t_k; \\ \mathcal{J}(m-1) - 1, & \text{if only corrupt nodes mine a block during round } t_k; \\ \mathcal{J}(m-1), & \text{otherwise.} \end{cases}$$

**Remark 21.** A couple of interesting facts on the coalescing opportunities dynamics are: Among the most recent m blocks in a longest chain, there are at least  $\mathcal{J}(m)$  blocks mined by the honest nodes. In addition, regardless of the behaviors of the adversary, for any two longest chains, there are at least  $\mathcal{J}(m)$  block positions each of which has non-zero probability of being in the common prefix of these two chains.

Let  $p_{+1} = \mathbb{P} \{ \mathcal{J}(m) = \mathcal{J}(m-1) + 1 \}$  and  $p_{-1} = \mathbb{P} \{ \mathcal{J}(m) = \mathcal{J}(m-1) - 1 \}$ , i.e.,  $p_{+1}$  (resp.  $p_{-1}$ ) is the probability for  $\mathcal{J}(m)$  to move up (resp. down) by 1. We have

$$p_{+1} = \frac{(1-p)^b \left(1 - (1-p)^{n-b}\right)}{1 - (1-p)^n} \text{ and } p_{-1} = \frac{\left(1 - (1-p)^b\right) (1-p)^{n-b}}{1 - (1-p)^n}.$$
 (2)

It is easy to see that when  $b > \frac{1}{2}n$ , it holds that  $p_{+1} > p_{-1}$ . For ease of exposition, let  $p^* = \mathbb{P}\{\mathcal{J}(t) \neq \mathcal{J}(t-1)\} = p_{+1} + p_{-1}$ .

**Lemma 22.** With probability at least  $\left(1 - \exp\left(-\frac{(p_{+1} - p_{-1})^2 M}{16p^*}\right) - \exp\left(-\frac{(p^*)^2 M}{2}\right)\right)$ , it holds that  $\mathcal{J}(M) \geq \frac{(p_{+1} - p_{-1})M}{4}$ .

Lemma 22 gives a high probability lower bound on the number of coalescing opportunities during M nonempty rounds. Its proof can be found in Appendix  $\mathbf{E}$ .

**Theorem 23.** For any given  $T \ge 1$  and  $M \ge \frac{4}{\beta(p_{+1}-p_{-1})}$  where  $\beta = \frac{(n-b)p}{2(3np)^2}$ , at the end of round T, with probability at least

$$1 - \exp\left(-\frac{(p^*)^2 M}{2}\right) - \exp\left(-\frac{(p_{+1} - p_{-1})^2 M}{16p^*}\right) - \frac{2}{\beta} \exp\left(-\frac{1}{2}(n - b)\right)$$

over the randomness in the block mining, the expected maximal inconsistency among a given pair of honest nodes is less than M, where the expectation is taken over the randomness in the symmetry breaking.

**Remark 24.** It is worth noting that  $\beta = \frac{(n-b)p}{2(3np)^2} = \frac{1}{18} \frac{(n-b)}{n} \frac{1}{np}$ , i.e.,  $\beta$  is a function of the fraction of honest nodes and the total mining power of the nodes in the system.

Suppose that  $n \geq 2\log\frac{4}{\epsilon\beta}$  for any given  $\epsilon \in (0,1)$ . Let

$$M^* = \max \left\{ \frac{4 \log 1/\epsilon}{(p^*)^2}, \frac{4}{\beta (p_{+1} - p_{-1})}, \frac{16p^*}{(p_{+1} - p_{-1})^2} \log \frac{4}{\epsilon} \right\}.$$

From Theorem 23, we know that with probability at least  $1 - \epsilon$ , the maximal inconsistency is less than  $M^*$ . Roughly speaking, when b gets smaller,  $M^*$  mainly gets smaller.

**Proof of Theorem 23.** We use  $N_t$  to denote the number of blocks generated during round t and associate each node with a distinct color in  $\{c_1, \dots, c_n\}$ . If node i mines a block during round t, we use  $(c_i, t)$  to denote this block. The genesis block is denoted as  $(c_1, 0)$ . Recall that the blocks mined during round t are collectively referred to as the block layer t. As the randomness in the block generation (i.e., puzzle solving of individual nodes) is independent of the adversarial behaviors of the corrupt nodes and is independent of which chain an honest node chooses to extend, we consider the auxiliary process wherein the nodes mine blocks for the first T rounds, and then the corrupt nodes and honest nodes sequentially decide on block attachments. Let  $\{i_1, \dots, i_K\}$  be the set of rounds such that  $N_{i_k} \neq 0$  for each  $i_k \in \{i_1, \dots, i_K\}$ . Let  $j_1$  and  $j_2$  be any two honest nodes whose chains at the end of round T are denoted by  $C_1(T)$  and  $C_2(T)$ , respectively. For each of these chains, we can read off a sequence of colors

for Chain 
$$C_1(T)$$
:  $c_1c(1,2)c(1,3)\cdots c(1,\ell_1)$ , and for Chain  $C_2(T)$ :  $c_1c(2,2)c(2,3)\cdots c(2,\ell_2)$ ,

where  $\ell_1$  and  $\ell_2$ , respectively, are the lengths of chains  $C_1(T)$  and  $C_2(T)$ ,  $c_1$  is the color of the genesis block, c(1,k) for  $k \in \{2, \dots, \ell_1\}$  is the color of the k-th block in  $C_1(T)$  and c(2,k) for  $k \in \{2, \dots, \ell_2\}$  is the color of the k-th block in  $C_2(T)$ . If  $\ell_1 \neq \ell_2$ , without loss of generality, we consider the case that  $\ell_1 < \ell_2$ ; the other case can be handled similarly. We augment the color sequence  $c_1c(1,2)c(1,3)\cdots c(1,\ell_1)$  to the length  $\ell_2$  sequence as

$$c_1c(1,2)c(1,3)\cdots c(1,\ell_1)c(1,\ell_1+1)\cdots c(1,\ell_2),$$

by setting  $c(1,k)=c_0$  for  $k=\ell_1+1,\cdots,\ell_2$  where  $c_0\notin\{c_1,\cdots,c_n\}$  is a special color that never shows up in a real block. It is easy to see that  $C_1(T)$  and  $C_2(T)$  start to be inconsistent at their k-th block if and only if  $c(1,k')\neq c(2,k')$  for each  $k'\in\{k,\cdots,\ell_2\}$ . Let  $\{i_{h_1},\cdots,i_{h_R}\}\subseteq\{i_1,\cdots,i_K\}$  such that for each  $i_{h_r}\in\{i_{h_1},\cdots,i_{h_R}\}$  it holds that

- Only honest nodes successfully mined blocks;
- $\mathcal{N}(i_{h_m-1}) = 0$ .

For ease of exposition, we refer to each of  $i_{h_r}$  as a coalescing opportunity. Recall that each of the honest nodes extends one of the longest chains it receives. By Lemma 19, we know that each of  $C_1(T)$  and  $C_2(T)$  contains a block generated during round  $i_{h_r}$ . Let  $(c'_1, i_{h_r})$  and  $(c'_2, i_{h_r})$  be the blocks included in  $C_1(T)$  and  $C_2(T)$ , respectively. If  $(c'_1, i_{h_r})$  is in the k-th position in  $C_1(T)$ , then  $(c'_2, i_{h_r})$  is also in the k-th position in  $C_2(T)$ . For each  $i_{h_r}$ , we denote the set of chains (including the forwarded chains) received by  $j_1$  and  $j_2$  at

round  $i_{h_r}$ , denoted by  $\mathcal{C}_1^r$  and  $\mathcal{C}_2^r$ . Since the adversary can hide chains to a selective group of honest nodes,  $\mathcal{C}_1^r$  and  $\mathcal{C}_2^r$  could be different. The probability of  $j_1$  and  $j_2$  extending the same chain at round  $i_{h_r}$  is

$$\frac{|\mathcal{C}_1^r \cap \mathcal{C}_2^r|}{|\mathcal{C}_1^r| \, |\mathcal{C}_2^r|} \ge \frac{\operatorname{NB}(i_{h_r-1})}{\left(\operatorname{NB}(i_{h_r-1}) + \operatorname{AB}(\widetilde{i_{h_r-1}}) + \operatorname{AB}(\widetilde{i_{h_r-1}})\right)^2}$$
(3)

where the inequality follows from Lemma 33. By Lemma 22, we know that in the M non-empty block layers that are most recent to round T,

$$R \ge \mathcal{J}(M) \ge \frac{(p_{+1} - p_{-1})M}{4}$$

holds with probability at least  $\left(1 - \exp\left(-\frac{(p^*)^2 M}{2}\right) - \exp\left(-\frac{(p_{+1} - p_{-1})^2 M}{16p^*}\right)\right)$ . In addition, it can be shown that for each of the r ensured by Lemma 22 we have

$$\max\{\left|\mathcal{C}_1^r\right|,\left|\mathcal{C}_2^r\right|\} \leq \mathrm{NB}(i_{h_r-1}) + \mathrm{AB}(i_{h_r-1}) + \mathrm{AB}(\widetilde{i_{h_r-1}})\mathbbm{1}\{\mathrm{AB}(i_{h_r-1}) = 0\}.$$

For any  $i_k$ , let  $X_k$  be the number of blocks mined by the honest nodes during round  $i_k$  such that  $X_k \neq 0$ . Using conditioning and Hoeffding's inequality, the following holds with probability at least  $(1-2\exp\left(-\frac{1}{2}(n-b)\right))$ ,

$$X_k \ge \frac{1}{2}(n-b)p$$
 and  $X_k + Y_k + Y_{k-1}\mathbb{1}\{Y_k = 0\} \le 3np$ ,

which implies that  $\frac{X_k}{X_k + Y_k + Y_{k-1} \mathbb{1}\{Y_k = 0\}} \ge \frac{(n-b)p}{2(3np)^2} \triangleq \beta$ . On average over the random symmetry breaking, it takes at most  $1/\beta$  coalescing opportunities backwards for chains  $C_1(T)$  and  $C_2(T)$  to coalesce into one. Thus, we need  $\frac{(p+1-p-1)M}{4} \ge \frac{1}{\beta}$ .

## 5 VDF-Based Scheme

In this section, we present a scheme to ensure Assumption 16. The key cryptographic tool we use in the following scheme is the construction of the *verifiable delay function*,  $\mathcal{F}(x)$ , which we define informally below. Please refer to [4] for the formal definition (also defined formally in the full version of our paper).

**Definition 25** (Verifiable Delay Function (informal)). Let  $\lambda$  be our security parameter. There exists a function  $\mathcal{F}$  with difficulty  $X = O(\text{poly}(\lambda))$  where the output  $y \leftarrow \mathcal{F}(x)$  (where  $x \in \{0,1\}^{\lambda}$ ) cannot be computed in less than X sequential computation steps, even provided  $\text{poly}(\lambda)$  parallel processors, with probability at least  $1 - \mathsf{negl}(\lambda)$ . The VDF output can be verified, quickly, in  $O(\log(X))$  time.

We set the difficulty of the VDF to the duration of a round; in other words, the difficulty is set such that the VDF produces exactly one output at the end of each round. We amend default Nakamoto consensus by adding the following procedure. We believe this could be added in a backwards-compatible way to existing Nakamoto implementations, like Bitcoin. Backwards-compatibility is desirable in decentralized networks because it means that a majority of the network can upgrade to the new protocol and non-upgraded nodes can still verify blocks and execute transactions. Below we describe a scheme that, when added to Nakomoto consensus, assures Assumption 16. The proof of the following theorem is in the full version of our paper.

**Theorem 26.** Assumption 16 is satisfied by our VDF-based scheme.

**VDF-Scheme Overview.** The VDF-scheme works intuitively as follows. We number the rounds beginning with round 0. All nodes have the genesis block  $B_0$  in their local chains in round 0 and starting mining blocks in round 1. In round 0, the VDF output is computed using 0 as the input. During each round j > 0, each node computes a VDF output,  $y_j$ , (using  $\mathcal{F}$ ) for the current round j where the input to  $\mathcal{F}$  is the output of the VDF,  $y_{j-1}$ , from the previous round concatenated with the round number, j. Both inputs are necessary; the

output of the VDF from the previous round ensures that we cannot compute the VDF output for this round until we have obtained the output for the previous round, and the round number is necessary to ensure that the output is not used for a future round. Once the VDF output is computed, each honest node attempts to mine a block using the VDF output as part of the input to the mining attempt. This also ensures that the block generation rate of honest nodes is upper bounded by np. Then, each node which successfully mines a block sends the new chain to all other nodes.

All honest nodes verify that each chain satisfies two conditions:

- 1. Let  $o_1, \ldots, o_\ell$  be the VDF outputs contained in blocks  $B_1, \ldots, B_\ell$ , respectively, of a chain C (the genesis block does not contain a VDF output). Let  $r_1, \ldots, r_\ell$  be the rounds where  $o_1, \ldots, o_\ell$  were computed, respectively. Then,  $r_1 < \cdots < r_{\ell-1} < r_\ell$ .
- 2.  $o_i$  is the VDF output computed from round  $r_i \geq i 1$ .

The honest nodes also check all proofs included in the chains, confirming that the VDF outputs are correctly computed and the blocks are correctly mined using the VDF outputs. An honest node discards any chain which does not pass verification.

**Pseudocode.** The precise pseudocode of our VDF-based scheme is given below. Using  $\mathcal{F}$ , each honest node i performs the following:

- 1. Initially, all honest nodes use input 0 at the start of the protocol to obtain output  $y_0 = \mathcal{F}(0)$  for round 0.
- 2. Let  $d_j = \mathcal{F}(y_{j-1})$  be the output of the VDF for round j and  $y_j = d_{j-1}|j.^6|i$  stores  $y_j$ .
- 3. When i mines a block  $B_j$ , i includes the output  $y_{j-1} = d_{j-1}|j$  from the previous round in  $B_j$ , ie.  $B_j$  is mined with  $y_{j-1}$  as part of the input.
- 4. Each node which successfully mines a block adds the mined block to its local chain. Then, it broadcasts its local chain to all other nodes.
- 5. For each longest chain received, each node verifies the following:
  - (a) Let  $o_1, \ldots, o_\ell$  be the VDF outputs stored in each block in order starting with the first block and ending with the  $\ell$ -th block. Let  $r_1, \ldots, r_\ell$  be the rounds associated with the VDF output. Then,  $r_\ell > r_{\ell-1} > \cdots > r_1$ .
  - (b) The k-th block in the chain (starting from the genesis block) is mined using  $y_{k'}$  from round  $k' \ge k 1$ .
  - (c) The proofs of the VDF output and the mining output are correct, i.e. the block is correctly mined using the corresponding VDF output.
- 6. If i receives a chain where more than one block in the chain is mined with the same  $y_j$  (for any j smaller than the current round), the node discards the chain.
- 7. At the end of round j, i sets  $y_{j+1} \leftarrow \mathcal{F}(y_j)|j+1$  and begins computing the next value  $\mathcal{F}(y_{j+1})$  using  $y_{j+1}$  as input.

Due to space constraints, we do not include the proof of Theorem 26; please find the full proofs in the full version of our paper. However, the intuition for our proof is straightforward. Items 5a and 5b ensure that no chain accepted by an honest node contains more than one block per VDF output. Setting the difficulty of the VDF to the duration of the round ensures that at most one VDF output is produced during a round. Together, these two observations prove Theorem 26, namely, that any chain held by an honest node can be extended by at most one block each round.

<sup>&</sup>lt;sup>6</sup>Here, a|b is the commonly used notation indicating concatenation between a and b.

## 6 Discussion

Validation and Communication Costs. A higher p means a faster block rate and thus more blocks. The validation and bandwidth complexity of Nakamoto protocols are proportional to block size and the number of blocks that are mined, since each miner validates and then communicates every mined block to all other miners (in practice, nodes do not necessarily gossip shorter chains, and taking advantage of nodes' memory overlap can help reduce block transfer size [8]). One needs to determine the optimal value of p that trades off validation and bandwidth complexity and chain growth. This work expands the space of p to consider.

Other Symmetry-Breaking Strategies. Here we consider three other symmetry-breaking strategies with high p. First-seen is where all honest nodes take the first chain out of the longest-length chains they see, and lexicographically-first is where honest nodes take the lexicographically-first chain of the set of longest chains according to some predetermined ordering, for example alphabetically. Intuitively, the adversary can control the network and thus cause different honest nodes to see different chains of the same length first for first-seen, impacting common-prefix, or grind on blocks to always produce the lowest lexicographically-ordered chain for lexicographically-first, impacting chain-quality. A third strategy is to use a global-random-coin: Suppose that all nodes have access to a permutation oracle  $\mathcal{P}$  that returns a permutation sampled uniformly at random of a number of elements passed into it where any subset of elements obey the same partial ordering. With  $\mathcal{P}$  symmetry-breaking is trivial since all honest nodes will agree on the result of the coin flip. Furthermore, if the coin is fair, then the number of honest blocks added to the chain is proportional to the fraction of honest nodes. However, in reality, it is difficult and oftentimes infeasible to ensure such a strong guarantee.

Conclusion. In this work we show that unlike previously thought, convergence opportunities are not necessary to make chain progress. We use  $coalescing\ random\ walks$  to analyze the correctness of Nakamoto consensus under a regime of puzzle difficulty previously thought to be untenable, expanding the space of p for protocol designers.

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# **Appendices**

# A Additional Definitions

#### A.1 VDFs

The formal definition of VDFs is presented below.

**Definition 27** (Verifiable Delay Functions [4]). A VDF V = (Setup, Eval, Verify) is a triple of algorithms that perform the following:

- 1. Setup(λ, X) → pp = (ek, vk): The Setup algorithm takes as input a security parameter λ and a desired difficulty level X and produces public parameters consisting of an evaluation key ek and a verification key vk. Setup is polynomial time with respect to λ and X is subexponentially-sized in terms of λ. The public parameters specify an input space X and an output space Y. X is efficiently sampleable. If secret randomness is used in Setup, a trusted setup might be necessary.
- 2. Eval(ek, x)  $\rightarrow$  (y,  $\pi$ ): Eval takes an input  $x \in \mathcal{X}$  (in the sample space of inputs) and the evaluation key and produces an output  $y \in \mathcal{Y}$  (in the sample space of outputs) and a (possibly empty) proof  $\pi$ . Eval may use random bits to generate  $\pi$  but not to compute y. Eval runs in parallel time X even when given  $poly(\log(X), \lambda)$  processors for all **pp** generated by Setup( $\lambda, X$ ) and  $x \in \mathcal{X}$ .
- 3.  $Verify(vk, x, y, \pi) \to \{Yes, No\}$ : Verify is a deterministic algorithm that takes the verification key vk, an input x, the output y, and proof  $\pi$  and outputs Yes or No depending on whether y was correctly computed from via Eval. Eval runs in time  $O(\log(X))$ .

Furthermore, V must satisfy the following properties:

- 1. Correctness A VDF V is correct if for all  $\lambda, X$ , parameters (ek, vk)  $\stackrel{R}{\leftarrow}$  Setup( $\lambda, X$ ), and all  $x \in \mathcal{X}$ , if  $(y, \pi) \stackrel{R}{\leftarrow}$  Eval(ek, x), then Verify(vk,  $x, y, \pi$ )  $\rightarrow Yes$ .
- 2. Soundness A VDF is sound if for all algorithms  $\mathcal{A}$  that run in time  $O(poly(X,\lambda))$

$$\mathbb{P}\left[\begin{array}{c} \mathtt{Verify}(\mathtt{vk},x,y,\pi) = Yes \\ y' \neq y \end{array} \middle| \begin{array}{c} \mathbf{pp} = (\mathtt{ek},\mathtt{vk}) \xleftarrow{R} \mathtt{Setup}(\lambda,X) \\ (x,y',\pi') \xleftarrow{R} \mathcal{A}(\lambda,\mathbf{pp},X) \\ (y,\pi) \xleftarrow{R} \mathtt{Eval}(\mathtt{ek},x) \end{array} \right] \leq negl(\lambda).$$

3. Sequentiality A VDF is  $(p, \sigma)$ -sequential if no adversary  $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$  with a pair of randomized algorithms  $\mathcal{A}_0$ , which runs in total time  $O(poly(X, \lambda))$ , and  $\mathcal{A}_1$ , which runs in parallel time  $\sigma(t)$  on at most p(t) processors, can win the following game with probability greater than  $negl(\lambda)$ :

$$\begin{aligned} \mathbf{pp} & \stackrel{R}{\leftarrow} \mathtt{Setup}(\lambda, X) \\ L & \stackrel{R}{\leftarrow} \mathcal{A}_0(\lambda, \mathbf{pp}, X) \\ x & \stackrel{R}{\leftarrow} \mathcal{X} \\ y_{\mathcal{A}} & \stackrel{R}{\leftarrow} \mathcal{A}_1(L, \mathbf{pp}, x). \end{aligned}$$

 $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$  wins the game if  $(y, \pi) \stackrel{R}{\leftarrow} \text{Eval}(ek, x)$  and  $y_{\mathcal{A}} = y$ .

There are many implementations in the literature of VDFs (e.g. [25, 28]). We do not provide these implementations here as it is out-of-scope for our paper, but please refer to these papers for contructions of VDFs that satisfy the above properties.

#### A.2 Tail Bounds

We use the following variant of Hoeffding's inequality.

**Theorem 28** (Hoeffding's inequality). Let  $Y_1, \ldots, Y_n$  be n independent, identically distributed random variables drawn from a Bernoulli distribution with parameter  $p, Y_i \stackrel{i.i.d.}{\sim} Ber(p)$ . If  $S_n = \sum_{i=1}^n Y_i$ , then  $S_n \sim Binom(n, p), \mathbb{E}(S_n) = np$ , and

$$\mathbb{P}\left\{\frac{1}{n}\left|\sum_{i=1}^{n}Y_{i}-pn\right|\geq\varepsilon\right\}\leq2e^{-2n\varepsilon^{2}}.$$

#### A.3 The Bitcoin Blockchain System

In this section, for completeness, we provide a high-level overview of the Bitcoin Blockchain System. The below is mainly to serve as a reminder of the Bitcoin protocol for those unfamiliar with it.

#### High-Level Description.

The nodes in the system represent miners in the Bitcoin cryptosystem who mine blocks filled with requests from *clients*. Clients represent payers who would like to fulfill some *transactions*. The client issues a *write-request* whenever it wants to send a transaction to a miner. The miner then attempts to *mine* a block containing the value of the transaction. Specifically, the following set of steps occur:

- The payer submits a write-request to the system with a valid transaction as the write "value" they want to add to the public ledger.
- Every honest miner i:
  - has a mempool which contains a collection of multi-cast transactions received by this miner.
     Notably, due to issues such as network failures and messages delay, the mempool kept by different miners might not be identical, and
  - keeps a local valid blockchain  $C_i$ .
- In each round, each of the miners:
  - 1. Blockify its local mempool (i.e., creates a block of appropriate size that contains a sub-set of the transactions in mempool) and removes those blockified transactions from mempool.
  - 2. Try to add this new block to its local chain  $C_i$ .
  - 3. If the miner successfully extends its local chain, it multi-casts the updated chain to other miners.
  - 4. Wait to receive multi-casted chains from others and update its local chain to be the chain that is the longest among the received chains and its current local chain. If there are multiple longest chains, use a **symmetry breaking** mechanism to choose one of them as its new local chain.

In the Bitcoin system, oftentimes, the symmetry is broken in an arbitrary manner, i.e., if there is a tie, an honest node chooses an arbitrary longest chain (e.g. the chain it received first). In an adversarial setting, this symmetry-breaking strategy could potentially lead to honest nodes choosing different chains frequently. It turns out that this symmetry-breaking rule, with high probability, can guarantee safety as long as it is sufficiently hard to successfully mine a block. However, this is not the case when the probability of successfully mining a block is large. In fact, for such instances, it is important to consider specific symmetry-breaking strategies and how they affect the system.

# B Honest Majority Assumption

The honest majority assumption in the seminal [13] is presented below for completeness. For ease of comparison, we use the same notation as that in [13] Let  $f_0$  be the probability at least one honest node succeeds in finding a proof-of-work (pow) in a round. In [13], the notion of the *advantage of honest participants* is used, denoted by  $\delta$ . It is used to bound  $\frac{b}{b-b}$ . In particular,  $\delta$  is chosen so that  $\frac{b}{b-b} \leq 1 - \delta$  always holds.

**Assumption 29** (Honest Majority Assumption [13]). Given an  $\epsilon \in (0,1)$ , n, and p, the maximal number of corrupted nodes b satisfies:

- $3f_0 + 3\epsilon < \delta \le 1$ , and
- $b \le (1 \delta)(n b)$ .

Notably, by definition of  $\delta$ , the second bullet in Assumption 29 always holds. Hence, for fixed  $\epsilon \in (0,1)$ , n, and p, the real constraint on b is the relation assumed in the first bullet of Assumption 29.

**Proof of Proposition 6.** By Assumption 29, it holds that

$$3f + 3\epsilon < \delta \le 1. \tag{4}$$

As  $\epsilon > 0$ , (4) implies that  $3f < \delta$ . Let f denote the probability at least one honest node succeeds in finding a pow in a round. We have  $f \ge 1 - (1 - p)^{(n-b)}$ . So

$$3\left(1 - \left(1 - p\right)^{(n-b)}\right) \le 3f < \delta.$$

Equivalently,

$$\log(1 - \delta/3) < (n - b)\log(1 - p),$$

for arbitrary base of log as long as the base is  $\geq 1$ . By Taylor expansion, we have

$$(n-b)p < \frac{\delta/3}{1-\delta/3} \le \frac{\delta}{2},$$

where the last inequality follows from the fact that  $\delta \in (0,1)$ .

On Remark 7: To see the claim in Remark 7, consider the boundary case where n = 2b + 1 – the honest nodes barely make it to be the majority of the system. In this case, the upper bound of p in Proposition 6 is

$$\frac{n-2b}{2(n-b)^2} = \frac{1}{2(n-\frac{n-1}{2})^2} = \frac{2}{(n+1)^2}.$$

Thus, in expectation, it takes at least n+1 rounds for the honest nodes to mine a block collectively. Such a low block generating speed makes it unlikely to have multiple longest chains unless the network delay is very serious. This observation also justifies why the choice of symmetry breaking rules does not matter much in [13,23].

This observation holds not only for the boundary case when n=2b+1 but also for more general b. For ease of illustration, let's consider the sequence of  $b_k$  for  $k=1,\cdots,\lfloor\frac{n-1}{2}\rfloor$  with  $b_k:=\lfloor\frac{k}{2k+1}n\rfloor$ . Without loss of generality, assume that  $\frac{k}{2k+1}n$  is an integer for all k under consideration. For a system with up to  $b_k$  corrupted nodes, the upper bound in Proposition 6 lies in between  $\left(\frac{1}{2(k+1)n}, \frac{1}{(k+1)n}\right)$ . In a sense, the Honest Majority Assumption (formally stated in 29) requires the mining puzzle becomes harder as  $b \to \frac{1}{2}n$ . That is, Assumption 29 requires the system to trade off liveness for tolerating more corrupt nodes.

## C Proof of Theorem 8

**Proof of Theorem 8.** We formalize the arguments of the main proof ideas in Section 4.1. Let  $\{c_1, \dots, c_n\}$  be a set of n different colors. We associate each node in the system with a color. We use  $(c_i, t)$  to denote the block generated by honest node i during round t and  $(c_0, 0)$  to denote the genesis block. We use  $(c_i, t) \to (c_{i'}, t-1)$  to denote the event that block  $(c_i, t)$  is attached to block  $(c_{i'}, t-1)$ , which occurs with probability  $\frac{1}{n}$  under our symmetry-breaking rule. To quantify the maximal inconsistency of the longest chains of round T, we consider the following auxiliary random process. It can be easily shown that there is a bijection between the sample paths of the Bitcoin blockchain protocol and the sample paths of this auxiliary process, and that the auxiliary process and the original blockchain protocol with random symmetry breaking have the same probability distribution.

Auxiliary random procedure: For any given  $T \geq 1$ , do the following:

- (i) Let each color generate a block for each of the rounds in  $\{1, 2, \cdots, T\}$ ;
- (ii) Attach each of the block  $(c_i, 1)$  for  $i = 1, \dots, n$  to the genesis block  $(c_0, 0)$ ;
- (iii) For each  $t \geq 2$  and each  $(c_i, t)$ , attach it to one of the blocks  $\{(c_i, t-1), i=1, \dots, n\}$  uniformly at random (i.e., with probability 1/n).

Connecting to coalescing random walks: Here, we formally quantify the connection between the maximal inconsistency among the longest chains of round T with the coalescing time of n random walks on an n-complete graph. Since p = 1 and there is no adversary, the number of longest chains received by each honest node at each round is n. Let  $C(T, c_1), \dots, C(T, c_n)$  be the n longest chains of round T ending with blocks  $(c_1, T), \dots, (c_n, T)$ , respectively. We first show that each of these n chains can be coupled with a random walk on the n-complete graph. Without loss of generality, let's consider  $C(T, c_1)$  which can be expanded as

$$C(T,c_1) := (c_0,0) \leftarrow (c_{i_1},1) \leftarrow \cdots \leftarrow (c_{i_{t-1}},t-1) \leftarrow (c_{i_t},t) \leftarrow \cdots \leftarrow (c_{i_{T-1}},T-1) \leftarrow (c_1,T), \tag{5}$$

where  $c_t$  is the color of the (t+1)-th block in the chain. Note that the chain  $C(T, c_1)$  is random because the sequence of block colors  $c_0c_{i_1}\cdots c_{i_{t-1}}c_{i_t}\cdots c_{i_{T-1}}c_1$  is random. Moreover, the randomness in  $C(T, c_1)$  is fully captured in the randomness of the block colors. We have

$$\mathbb{P}\left\{C(T,c_1) = (c_0,0) \leftarrow (c_{i_1},1) \leftarrow \cdots \leftarrow \left(c_{i_{t-1}},t-1\right) \leftarrow (c_{i_t},t) \leftarrow \cdots \leftarrow \left(c_{i_{T-1}},T-1\right) \leftarrow (c_1,T)\right\} \\
\stackrel{(a)}{=} \mathbb{P}\left\{(c_0,0) \leftarrow (c_{i_1},1)\right\} \prod_{t=2}^T \mathbb{P}\left\{\left(c_{i_{t-1}},t-1\right) \leftarrow (c_{i_t},t)\right\} \\
= \prod_{t=2}^T \mathbb{P}\left\{\left(c_{i_{t-1}},t-1\right) \leftarrow (c_{i_t},t)\right\},$$

where the last equality is true as  $\mathbb{P}\{(c_0,0)\leftarrow(c_{i_1},1)\}=1$ , and the equality (a) holds because under our symmetry-breaking rule, neither the previous history up to round t nor the future block attachment choices after round t affects the choice of the chain extension in round t. Moreover, the probability of any realization of the color sequence  $c_0c_{i_1}\cdots c_{i_{t-1}}c_{i_t}\cdots c_{i_{t-1}}c_1$  (i.e., a sample path on the block colors in Bitcoin) is  $\left(\frac{1}{n}\right)^{T-1}$ . Let's consider the complete graph with vertex set  $\{c_1,c_2,\cdots,c_n\}$ . Under our symmetry breaking rule, the backwards color sequence  $c_1c_{i_{T-1}}\cdots c_{i_t}c_{i_{t-1}}\cdots c_{i_1}$  (without considering the genesis block) is a random walk on the n-complete graph starting at vertex  $c_1$ . Similarly, we can argue that  $C(T,c_2),\cdots,C(T,c_n)$  correspond to n-1 random walks on the n-complete graphs starting at vertices  $c_2,\cdots,c_n$ , respectively. As argued in the main proof ideas paragraph, these n random walks are not fully independent. In fact, they are coalescing random walks, and their coalescence is exactly the maximal inconsistency among the longest chains of round T

With the above connection of the longest chain protocol augmented by uniformly-at-random symmetry breaking with coalescing random walks. We conclude by applying Theorem 11.

# D Missing proofs and auxiliary results for Section 4.2

Proof of Lemma 15. To characterize the coalescence time, similar to the analysis in [6], for any given  $k \in \{1, \dots, n_g\}$ , we construct a larger graph  $Q = Q_k = (V_Q, E_Q)$ , where  $V_Q = V^k$  and two vertices  $\boldsymbol{v}, \boldsymbol{w} \in V^k$  if  $\{v_1, w_1\}, \dots, \{v_k, w_k\}$  are edges of G. Let  $M_k$  be the time until the first meeting in the original graph G. Let  $S \subseteq V_Q$  denote the set of all possible configurations of the locations of the  $n_g$  random walks at the first meeting,

$$S_k = \{(v_1, \dots, v_k) : v_i = v_j \text{ for some } 1 \le i < j \le k\}.$$
 (6)

It is easy to see that there is a direct equivalence between the u-lazy random walks on G and the single u-lazy random walk on Q. Since Q is a complete graph with self-loops, the limiting distribution of lazy random walk on Q is the same as the standard random walk on Q. Let  $\boldsymbol{\pi}^Q \in \mathbb{R}^{|V^k|}$  be the stationary distribution of a standard random walk on Q and let  $\pi_{S_k}^Q = \sum_{\boldsymbol{v} \in S_k} \pi_{\boldsymbol{v}}^Q$ . By [6, Lemma 4], we know that for any  $1 \le k \le k^*$  where  $k^* \triangleq \max\{2, \log n_q\}$ , it holds that

$$\pi_{S_k}^Q \ge \frac{k^2}{8n_q}.$$

Let  $H_{\boldsymbol{v},S_k}$  denote the hitting time of vertex set  $S_k$  starting from vertex v and let

$$H^Q_{oldsymbol{\pi}}(H_{S_k}) = \sum_{oldsymbol{v} \in V^k} oldsymbol{\pi}^Q_{oldsymbol{v}} H_{oldsymbol{v},S_k}$$

denote the expected hitting time of  $S_k$  from the stationary distribution  $\pi^Q$ . From [1, Lemma 2.1] and the fact we can contract the vertex set  $S_k$  into one pseudo vertex, similar to [6, proof of Theorem 2], we have that

$$\mathbb{E}_{\boldsymbol{\pi}^{Q}}[H_{S_{k}}] = \frac{\sum_{t=0}^{\infty} \left( P_{S_{k}}^{t}(S_{k}) - \boldsymbol{\pi}_{S_{t}}^{Q} \right)}{\boldsymbol{\pi}_{S_{k}}^{Q}} = \frac{\sum_{t=0}^{\infty} \left( (1-u)^{t} + (1-(1-u)^{t}) \, \boldsymbol{\pi}_{S_{k}}^{Q} - \boldsymbol{\pi}_{S_{k}}^{Q} \right)}{\boldsymbol{\pi}_{S_{k}}^{Q}}$$
$$\leq \frac{8n_{g}}{k^{2}} \frac{1}{u} \left( 1 - \boldsymbol{\pi}_{S_{k}}^{Q} \right) \leq \frac{8n_{g}}{uk^{2}}.$$

In addition, by conditioning on whether the particles stay at their initial locations or not, we have

$$\mathbb{E}[M_k] = (1 - u)(1 + \mathbb{E}[M_k]) + u(1 + \mathbb{E}_{\pi^Q}(H_{S_k})),$$

which implies that

$$\mathbb{E}\left[M_k\right] \le \frac{1}{u} \left(1 + \frac{8n_g}{k^2}\right) = O\left(\frac{n_g}{uk^2}\right).$$

Thus, for any k such that  $1 \le k \le k^* = \{2, \log n_g\}$ , we have

$$\mathbb{E}\left[C_k\right] \le \sum_{s=2}^k \mathbb{E}\left[M_s\right] \le O(n_g/u).$$

Let  $\mathcal{W}_u$  be a lazy random walk on the complete graph G with initial location u. In each round, with probability (1-u),  $\mathcal{W}_u$  stays at its current location and with probability u it moves to one of the current neighbors (including self-loops) uniformly at random. Let  $\pi^G$  the limiting distribution of the location vertex of  $\mathcal{W}_u$ . By [6, Eq.(8)], its mixing time is  $t_{mix} = \frac{3 \log n_g}{\log(1/(1-u))}$ , i.e., for any given  $u \in V$ , when  $t \geq \lceil \frac{3 \log n_g}{\log(1/(1-u))} \rceil$ ,

$$\begin{split} \|P_u^t - \pmb{\pi}^G\|_1 &= \sum_{v \in V} \left| P_u^t(v) - \pmb{\pi}_v^G \right| \\ &= \left| 1 - \pi_u^G \right| (1 - u)^t + \sum_{v: v \in V, v \neq u} \left| \left( 1 - (1 - u)^t \right) \pmb{\pi}_v^G - \pmb{\pi}_v^G \right| \\ &\leq 2(1 - u)^t \leq \frac{2}{n_g^3} \leq \frac{1}{n_g^2}. \end{split}$$

Here, with a little abuse of notation, we use  $P_u^t$  to denote the distribution of the state of  $\mathcal{W}_u$  at round t. Let  $t^* = k^* \log n_q \left( k^* t_{mix} + 3\mathbb{E}_{\pi^Q} \left( H_{S_{k^*}} \right) \right)$ . Following the arguments in [6, Section 5], we have

$$\begin{split} C(n_g) & \leq 4t^* + \mathbb{E}\left[C_{k^*}\right] \\ & \leq 4\log n_g \left(k^* t_{mix} + 3\mathbb{E}_{\pi^Q}\left(H_{S_{k^*}}\right)\right) + O(n_g/u) \\ & \leq \frac{4\log^4 n_g}{\log\frac{1}{1-u}} + 12\log^2 n_g \frac{8n_g}{u\log^2 n_g} + O(n_g/u) \\ & \leq \frac{4\log^4 n_g}{u} + \frac{96n_g}{u} + O(n_g/u) \\ & = O(n_g/u), \end{split}$$

where the last inequality follows from  $\log 1/(1-u) \ge u$ .

The following lemma will be used in the proof of Theorem 12

**Lemma 30.** Suppose that there k balls. Let X be the number of non-empty bins if we throw each of the k balls into b bins, where  $k \leq b$ , uniformly at random. Let  $\tilde{X}$  be the number of non-empty bins if we throw each of the k balls into  $b + \Delta$  bins, where  $\Delta \in \mathbb{N}$ , uniformly at random. Then  $\tilde{X}$  first-order stochastically dominates X.

$$\mathbb{P}\left\{X\leq l\right\}\geq \mathbb{P}\left\{\tilde{X}\leq l\right\} \quad \forall l.$$

*Proof.* Intuitively speaking, since  $b < b + \Delta$ , collisions are more likely to occur when fewer bins are available. Hence,  $X_2$  first-order stochastically dominates  $X_1$ . For the sake of peace of mind, a formal proof is given below.

Let's consider the mental process wherein we throw the balls into bins one by one. Let  $Y_t$  after we throw t balls into b bins. Similarly,  $Z_t$  be the number of nonempty bins we throw t balls into  $b + \Delta$  bins. We show Lemma 30 by induction on t.

Clearly,  $Y_1 = 1 = Z_1$ .

Induction hypothesis: Suppose for  $t \leq k-1$ , there exists a coupling between the marginal probabilities of the above two ball throwing processes such that under this coupling

$$Y_t \le Z_t. \tag{7}$$

When  $Y_t \leq Z_t - 1$ , by Eq.(7) and the monotonicity of Y and Z, it holds that  $Y_{t+1} \leq Y_t + 1 \leq Z_t \leq Z_{t+1}$ . It remains to consider the case where  $Y_t = Z_t$ . It is easy to see that  $Y_{t+1} = Y_t$  if the t+1-th was thrown into the existing non-empty bins, which occurs with probability  $\mathbb{P}\{Y_{t+1} = Y_t\} = \frac{Y_t}{b}$ . Similarly,  $\mathbb{P}\{Z_{t+1} = Z_t\} = \frac{Z_t}{b+\Delta}$ . For ease of exposition, let  $Z_{t+1} = Z_t = \gamma$ . Consider the following coupling:

- If the (t+1)-th ball of the second bins-and-balls process is thrown into the  $Z_t$  existing nonempty bins, then put the (t+1)-th ball of the first bins-and-balls process uniformly at random into its  $Y_t$  existing nonempty bins.
- If the (t+1)-th ball of the second bins-and-balls process is thrown into an empty bin, then with probability  $\frac{\Delta\gamma}{b(b+\Delta-\gamma)}$  put the (t+1)-th ball of the first bins-and-balls process uniformly at random into one existing nonempty bin. With probability  $1 \frac{\Delta\gamma}{b(b+\Delta-\gamma)}$ , put the (t+1)-th ball of the first bins-and-balls process into one empty bin uniformly at random.

It is easy to see that in the above coupling, the (t+1)-th ball of the first bins-and-balls process is throw into a bin (regardless whether it is empty or not) with probability  $\frac{1}{b}$ . Moreover, with this coupling and the induction hypothesis, we know that

$$Y_{t+1} \leq Z_{t+1}$$
,

completing the induction proof. Hence,  $X = Y_k \le Z_k = \tilde{X}$ . Therefore,

$$\mathbb{P}\left\{\tilde{X} \leq l\right\} \leq \mathbb{P}\left\{X \leq l\right\}, \ \forall \, l,$$

i.e.,  $\tilde{X}$  first-order stochastically dominates X.

**Proof of Theorem 12.** For any t, the expected length of a longest chain is  $1 + (1 - (1 - p)^n)t$ . When  $p < \frac{4 \ln 2}{n}$ , we can use Poisson approximation to approximate the distribution of number of blocks in each round. A straightforward calculation shows that the probability of having exactly one block in a round is  $np \exp(-np)$ . Thus, in expectation, the maximal inconsistency is at most  $\frac{1}{np \exp(-np)}$ . Henceforth, we restrict our attention to the setting where  $p \ge \frac{4 \ln 2}{n}$  and quantify the expected maximal inconsistency among the longest chains of round t. We first consider a coarse analysis whose arguments are similar to the proof of Theorem 8 and derive a bound on the maximal inconsistency via stochastic dominance. Though the obtained bound could be very loose, based on the insights obtained in this coarse analysis, we can come up with a much fine-grained analysis, which significantly improves the bound on maximal inconsistency.

A coarse analysis: Let  $\{c_1, \dots, c_n\}$  be a set of n different colors. We temporarily associate each node in the system with a color. <sup>7</sup> If node i mines a block during round t, we denote this block by  $(c_i, t)$ . In addition, we use  $(c_0, 0)$  to denote the genesis block. We use  $(c_i, t) \to (c_{i'}, t - 1)$  to denote the event that both blocks  $(c_i, t)$  and  $(c_{i'}, t - 1)$  exist and that block  $(c_i, t)$  is attached to block  $(c_{i'}, t - 1)$ , which, under our symmetry-breaking rule, occurs with probability

$$\frac{\mathbb{1}\{\text{node }c_i \text{ mines a block during round }t\}\mathbb{1}\{\text{node }c_{i'} \text{ mines a block during round }t-1\}}{\sum_{i'=1}^n\mathbb{1}\{\text{node }c_{i'} \text{ mines a block during round }t-1\}}.$$

Notably, in the Bitcoin protocol, there are two sources of randomness: (1) the randomness in generating blocks and (2) the randomness in the block attachments. To quantify the maximal inconsistency of the longest chains of round T, we consider the following auxiliary random process. It can be easily shown that there is a bijection between the sample paths of the Bitcoin blockchain protocol and the sample paths of this auxiliary process, and that the auxiliary process and the original blockchain protocol with random symmetry breaking have the same probability distribution.

Auxiliary random procedure: For any given  $T \geq 1$ , do the following:

- (i) For each of the rounds in  $\{1, 2, \dots, T\}$ , let each node/color generate a block with probability p independently of other nodes and independently across rounds. For ease of exposition, we refer to the blocks mined in round t as the blocks in layer t.
- (ii) Attach each of the block  $(c_i, 1)$ , if exists, for  $i = 1, \dots, n$  to the genesis block  $(c_0, 0)$ ;
- (iii) For each  $t \ge 2$  and each  $(c_i, t)$  that exits, attach it to one of the blocks in layer (t 1). If layer (t 1) is empty, let

$$t' \triangleq \max\{r : \text{layer } r \text{ is nonempty and } r < t\}$$
,

and let each existing  $(c_i, t)$  uniformly at random chooses one ancestor block in block layer t'.

Connecting to coalescing random walks: We first build a coarse connection between the maximal inconsistency among the longest chains of round T with the coalescing time of n random walks on an n-complete graph. A much fine-grained connection to coalescing random walks on 2np-complete graph in given in fine-grained analysis part of this proof. It is easy to see that the number of blocks mined in each round t, denoted by  $N_t$ , follows the Binom(n,p) distribution. Without loss of generality, we assume that  $N_T \neq 0$ . If this does not hold, then we can replace T by the most recent round T' such that  $N_{T'} \neq 0$  and the remaining proof goes through. Since there is no adversary, the number of longest chains at the end of round T is  $N_T$ , each of which ends with a block in block layer T. Let  $C(T, c'_1), \dots, C(T, c'_{N_T})$  be the  $N_T$  longest chains of round T ending with blocks  $(c'_1, T), \dots, (c'_{N_T}, T)$ , respectively. We first show that each of these  $N_T$  chains

<sup>&</sup>lt;sup>7</sup>In our fine-grained analysis, the color of a block will be re-assigned.

can be coupled with a process that is a variant of a random walk on the *n*-complete graph. Without loss of generality, let's consider  $C(T, c'_1)$  which can be expanded as

$$C(T, c_1') := (c_0, 0) \leftarrow (c_{i_1}, 1) \leftarrow \cdots \leftarrow (c_{i_{k-1}}, k-1) \leftarrow (c_{i_k}, k) \leftarrow \cdots \leftarrow (c_{i_{K-1}}, K-1) \leftarrow (c_1', T), \tag{8}$$

where  $c_k'$  is the color of the (k+1)-th block in the chain and K is the number of non-empty block layers under event E – in the realization of block mining. Recall that for general  $p \in (0,1)$  there are two sources of randomness (1) the randomness in block generating and (2) the randomness in block attachment. Consequently, the sequence of block colors  $c_0c_{i_1}\cdots c_{i_{k-1}}c_{i_k}\cdots c_{i_{K-1}}c_1'$  is random in that, roughly speaking, the "feasibility" of  $c_{i_k}$  is determined by whether node  $i_k$  mines a block during round k or not, and the ordering of the "feasible" colors is determined by the attachment choices. Let E be any realization of the block mining for the first T rounds, which corresponds to any realization of step (i) of the auxiliary process. We have

$$\mathbb{P}\left\{C(T, c_{1}') \mid E\right\} \\
= \mathbb{P}\left\{(c_{0}, 0) \leftarrow (c_{i_{1}}, 1) \leftarrow \cdots \leftarrow (c_{i_{k-1}}, k-1) \leftarrow (c_{i_{k}}, k) \leftarrow \cdots \leftarrow (c_{i_{K-1}}, K-1) \leftarrow (c_{1}', T) \mid E\right\} \\
\stackrel{(a)}{=} \mathbb{P}\left\{(c_{0}, 0) \leftarrow (c_{i_{1}}, 1) \mid E\right\} \left(\prod_{k=2}^{K-1} \mathbb{P}\left\{(c_{i_{k-1}}, k-1) \leftarrow (c_{i_{k}}, k) \mid E\right\}\right) \mathbb{P}\left\{(c_{i_{K-1}}, K-1) \leftarrow (c_{1}', T) \mid E\right\} \\
= \left(\prod_{k=2}^{K-1} \mathbb{P}\left\{(c_{i_{k-1}}, k-1) \leftarrow (c_{i_{k}}, k) \mid E\right\}\right) \mathbb{P}\left\{(c_{i_{K-1}}, K-1) \leftarrow (c_{1}', T) \mid E\right\},$$

where the last equality is true as  $\mathbb{P}\{(c_0,0)\leftarrow(c_{i_1},1)\mid E\}=1$ , and the equality (a) holds because under our symmetry-breaking rule, neither the previous history up to round t nor the future block attachment choices after round t affects the choice of the chain extension in round t. Moreover, the conditional probability of any realization of the color sequence  $c_0c_{i_1}\cdots c_{i_{k-1}}c_{i_k}\cdots c_{i_{K-1}}c_1'$  conditioning on E (i.e., a sample path on the block colors in Bitcoin) is  $\prod_{t:2\leq t\leq T,\&n_t\neq 0}\frac{1}{n_t}$  where  $N_t=n_t$  for all  $t\in\{2,\cdots,T\}$ . Let's consider the complete graph with vertex set  $\{c_1,c_2,\cdots,c_n\}$ . Under our symmetry breaking rule, conditioning on E, the backwards color sequence  $c_1'c_{i_{K-1}}\cdots c_{i_k}c_{i_{k-1}}\cdots c_{i_1}$  (without considering the genesis block) is a walk, though not the standard random walk, of length T on the n-complete graph with initial location  $c'_1$ . Similarly, we can argue that  $C(T, c'_2), \dots, C(T, c'_{N_T})$  correspond to  $(N_T - 1)$  walks on the *n*-complete graphs starting at vertices  $c'_2, \dots, c'_{N_T}$ , respectively. Similar to the argument in the proof of Theorem 8, conditioning on E, there is an one-to-one correspondence between the event of the forking of the chains  $C(T, c_1'), \cdots, C(T, c_{N_x}')$ and the event of coalescence of the backwards walks; that is, the maximal inconsistency of the longest chains of round T is the same as the coalescence time of the  $N_T$  walks on the n-complete graphs. These  $N_T$  walks are more likely to coalesce than the standard random walks whose transition probability is 1/n, whereas under any  $E, N_t = n_t \le n$  for  $t = 1, \dots, T$ ; this fact can be formally shown via Lemma 30. Hence, the conditioning on E, the maximal inconsistency is upper bounded by the coalescence time of the standard random walks on n-complete graph with n particles, which is O(n). Since this is true for all possible block mining realization E, we conclude that the maximal inconsistency is upper bounded by O(n).

A fine-grained analysis: Let E any realization of the block mining for the first T rounds. If not explicitly mentioned, the following arguments are stated conditioning on E. To conclude the proof, towards the end of this proof, we take average over all possible events E.

The bound on the maximal inconsistency is O(n) which could be loose for a wide value range of p. This is because the upper bound on  $N_t$  is loose. Recall that  $N_t$  is a Binom(n,p). Thus,  $\mathbb{E}\left[N_t\right] = np \ll n$  as long as p = o(1). Observing this, in this fine-grained analysis, we first construct a lazy version of the  $N_T$  backwards walks whose expected coalescence time is at least the coalescence time of the original  $N_T$  backwards walks. Then re-color the mined blocks so that the re-colored lazy version of the  $N_T$  original walks are walks on at most 2np colors only. Then, we connect these lazy walks with the a lazy version of random walks each of which, if not stay at their current locations concurrently, moves to one of the neighboring colors (including its current color) with probability  $\frac{1}{2np}$ . Finally, by changing of the order of taking expectation, we show that the maximal inconsistency is upper bounded by the expected coalescence time of the  $(1-2\exp\left(-\frac{1}{3}np\right))$ -lazy random walks. We conclude the proof of this theorem by applying Lemma 15.

<u>Lazy walks construction</u>: Consider the following  $n_T$  lazy version of the backwards coalescing walks. For each  $s = 1, \dots, K - 1$ , if  $n_{K-s} \leq 2np$ , each of the remaining walks moves to one of the block color in layer

(K-s) uniformly at random. If two or more walks visit the same color, then these walks coalesce into one. If  $n_{K-s} > 2np$ , we let the remaining walks stay at their current color vertices. Clearly, this lazy version of the  $N_T$  backwards walks are more likely to coalesce than the original  $N_T$  walks. Let  $C_l(E)$  denote the expected number of backwards steps until all the lazy  $N_T$  walks coalesce.

Color re-assignment: Next we show that, under any E,  $C_l(E)$  is upper bounded by the expected coalescence time of  $\max\{2np,n_T\}$  lazy random walks on the 2np-complete graph. Towards this, we first do color-reassignment, detailed as follows. Let  $\{\tilde{c}_1,\cdots,\tilde{c}_n\}$  be a set of n different colors. We (re-)assign a color to each of the mined block in different block layers as follows: Assign color  $\tilde{c}_1$  to the genesis block. For each  $t=1,\cdots,T$  such that  $N_t\neq 0$ , let  $i_1^t<\cdots< i_{N_t}^t$  be the indices of the nodes/original colors each of which successfully mines a block during round t. Re-assign colors  $\tilde{c}_1,\cdots,\tilde{c}_{N_t}$  to blocks  $\left(c_{i_1^t},t\right),\cdots,\left(c_{i_{N_t}^t},t\right)$ . The re-colored blocks are denoted as  $(\tilde{c}_1,t),\cdots,(\tilde{c}_{N_t},t)$ , respectively.

Notably, different from the color assignment we used in the proof for the case when p=1, under the above color re-assignment rule, the blocks mined by the same node at different rounds could be assigned different colors. Fortunately, it is easy to see that the blocks attachments are independent of the color assignments. In particular, it is still true that the maximal inconsistency among the longest chains of round T is the same as the coalescence time of the corresponding backwards walks on those colors. Moreover, it is still true that the expected coalescence time of the re-colored  $N_T$  walks is upper bounded by  $C_l(E)$  the lazy version of the re-colored  $N_T$  walks. More importantly, the re-colored lazy  $N_T$  walks, expect for their initial colors, are the walks on at most 2np colors in each round.

Consider the following max  $\{2np, N_T\}$  lazy coalescing random walks on the 2np-complete graph with arbitrary but distinct initial locations. For each s=1,...T, if the above  $N_T$  lazy version of the walks on colors stay at their own locations concurrently (i.e.,  $N_{T-s} \geq 2np$ ) or  $N_{T-s} = 0$ , then each of the remaining random walks on the 2np-complete graph also stay at their current locations. If otherwise (i.e., each of the lazy walks on colors moves to one of the colors assigned to the blocks in the proceeding non-empty layer uniformly at random), we let each of the remaining walks on the 2np-complete graph moves to one of the 2np vertices uniformly at random. By Lemma 30, we know the expected coalescence time of the  $N_T$  lazy walks on colors is upper bounded by that of the max  $\{2np, N_T\}$  walks on the 2np-complete graph which is again upper bounded by the expected coalescence time of 2np lazy walks on the 2np-complete graph, denoted by  $C_{l,np}(E)$ .

Next we consider averaging over all the realizations of E. For any given T, with the above arguments, we know that the expected maximal inconsistency is upper bounded by

$$\sum_{E} C_{l,np}(E) \mathbb{P} \{E\}.$$

Note that by construction,  $C_{l,np}(E)$  depends on E only through the number of blocks mined in each round. In particular, it only depends on each  $N_t$  in whether  $1 \le N_t < 2np$  holds or not. So we have

$$\begin{split} &\sum_{E} C_{l,np}(E) \mathbb{P} \left\{ E \right\} = \sum_{n_{1}, \cdots, n_{T-1}} C_{l,np}(n_{1}, \cdots, n_{T-1}) \mathbb{P} \left\{ N_{t} = n_{t}, \forall \ t \leq T - 1 \right\} \\ &= \sum_{n_{1}, \cdots, n_{T-2}} \left( \sum_{n_{T-1}} C_{l,np}(n_{1}, \cdots, n_{T-1}) \mathbb{P} \left\{ N_{T-1} = n_{T-1} \right\} \right) \mathbb{P} \left\{ N_{t} = n_{t}, \forall \ t \leq T - 2 \right\} \\ &= \sum_{n_{1}, \cdots, n_{T-2}} \left( C_{l,np}(n_{1}, \cdots, n_{T-1} \in \{2np, \cdots, n\} \cup \{0\}) \mathbb{P} \left\{ N_{T-1} \geq 2np \text{ or } N_{T-1} = 0 \right\} \\ &+ C_{l,np}(n_{1}, \cdots, n_{T-1} \in \{1, \cdots, 2np - 1\}) \mathbb{P} \left\{ 1 \leq N_{T-1} < 2np \right\} \right) \mathbb{P} \left\{ N_{t} = n_{t}, \forall \ t \leq T - 2 \right\}. \end{split}$$

Note that when  $n_{T-1} \in \{2np, \dots, n\} \cup \{0\}$ , each of the 2np random walks stay at their initial locations concurrently with occurs with probability  $\mathbb{P}\{N_{T-1} \geq 2np \text{ or } N_{T-1} = 0\}$ , and when  $n_{T-1} \in \{1, \dots, 2np-1\}$ , each of the 2np random walks take one step standard coalescing random walks. That is, the 2np random walks are performing one step  $(1 - \mathbb{P}\{N_{T-1} \geq 2np \text{ or } N_{T-1} = 0\})$ -lazy random walk on the 2np-complete graph. Since  $N_t$  is i.i.d. across t, we can repeat this argument for T-1 times. In fact, we can exchange the order taking expectation over E and taking expectation over the realization of the walks. Hence,

 $\sum_{E} C_{l,np}(E) \mathbb{P}\{E\}$  equals the expected coalescence time of 2np  $(1 - \mathbb{P}\{N_{T-1} \ge 2np \text{ or } N_{T-1} = 0\})$ -lazy random walks. By Lemma 15, we know that

$$\sum_{E} C_{l,np}(E) \mathbb{P} \{ E \} = O \left( \frac{2np}{(1 - \mathbb{P} \{ N_{T-1} \ge 2np \text{ or } N_{T-1} = 0 \})} \right).$$

In addition, we have

$$\begin{split} \mathbb{P}\left\{N_t \geq 2np, \text{ or } N_t = 0\right\} &= \mathbb{P}\left\{N_t \geq 2np\right\} + \mathbb{P}\left\{N_t = 0\right\} \\ &= \mathbb{P}\left\{N_t \geq 2np\right\} + (1-p)^n = \mathbb{P}\left\{N_t \geq 2np\right\} + \exp\left(-n\log\frac{1}{1-p}\right) \\ &\leq \exp\left(-\frac{1}{3}np\right) + \exp\left(-n\log\frac{1}{1-p}\right) \leq 2\exp\left(-\frac{1}{3}np\right), \end{split}$$

where the last inequality holds because  $p \leq \log \frac{1}{1-p}$  when  $p \in [0,1)$ . So, it holds that

$$\sum_{E} C_{l,np}(E) \mathbb{P} \{E\} = O\left(\frac{2np}{\left(1 - 2\exp\left(-\frac{1}{3}np\right)\right)}\right),$$

proving the theorem.

E Missing proofs and auxiliary results for Section 4.3

**Proof of Lemma 19.** For each round  $\tau$ , the following holds:

- If no blocks are mined, then the lengths of the adversarial longest chains and the honest longest chains (longest chains kept by an honest node) are not changed.
- If both the honest and the corrupt nodes mine a block, then the lengths of the adversarial longest chains increases by 1, and length of the honest longest chains (longest chains kept by an honest node) increases by at least 1. To see the later, let's denote the length of the longest chains at the honest nodes at round  $(\tau 1)$  by  $\ell_{(\tau 1)}$  and the length of the longest chains at the honest nodes at round  $\tau$  by  $\ell_{\tau}$ . By the selective relay rule specified right before Definition 17, by the beginning of round  $\tau$ , every honest node has received a chain that is at least  $\ell_{(\tau 1)}$ . If the adversary does not release a prefix of an adversarial chain of length  $> \ell_{(\tau 1)}$ , the length of the longest chains kept by the honest nodes at round  $\tau$  is  $\ell_{\tau} = \ell_{(\tau 1)} + 1$ . Otherwise, due to the longest chain policy, it holds that  $\ell_{\tau} > \ell_{(\tau 1)} + 1$ .
- If only corrupt nodes mine a block, then the adversary can grow the length of the adversarial longest chains by 1. The length of the longest chains at the honest nodes is unchanged.
- If only honest nodes mine a block, then the length of the honest longest chains increase by at least 1. The formal argument follows the same as the proof of the later part of the second bullet. The length of the longest chains at the corrupt nodes (adversary) is unchanged.

Let  $t' \triangleq \max\{t' : \mathcal{N}(t') = 0 \text{ and } t' \leq t\}$ . Let  $t' = t_0 < t_1 < \dots < t_k$  be the round indices of the jumps of the random process  $\mathcal{N}(\tau)_{\tau=0}^{\infty}$ , i.e.,

$$\mathcal{N}(t_0) \neq \mathcal{N}(t_0+1), \cdots, \mathcal{N}(t_k-1) \neq \mathcal{N}(t_k).$$

By the construction of  $\mathcal{N}(\tau)_{\tau=0}^{\infty}$ , we know that from round t'+1 to round t, the number of rounds in which only corrupt nodes mine a block is  $\mathcal{N}(t)$  larger than the number of rounds in which only honest nodes mine a block.

Therefore, we know that as long as at round t', the length of the adversarial longest chain is no longer than the length of the honest longest chains, we can conclude that at the end of round t, the length of the

adversarial longest chains is at most  $\mathcal{N}(t)$  blocks longer than the length of the honest longest chains. It remains to show the following is true: "At round t', the length of the adversarial longest chain is no longer than the length of the honest longest chain."

Let  $\mathcal{N}(t')$  be the k-th time starting from round 0 such that

$$\mathcal{N}(t') = 0$$
 and  $\mathcal{N}(t'+1) = 1$ .

If k = 1, by the arguments in the above four bullets, we know the claim holds. Let's assume this claim holds for general r. We next prove it holds for r + 1. Let t'' be the r-th time starting from round 0 such that

$$\mathcal{N}(t'') = 0$$
 and  $\mathcal{N}(t'' + 1) = 1$ .

By induction hypothesis, we know that at round t'', the length of the adversarial longest chain is no longer than the length of the honest longest chain. By the first part of the proof of Lemma 19, we know at round t', the length of the adversarial longest chain is no longer than the length of the honest longest chain. Thus, the proof of the claim is complete.

The following lemma follows from Hoeffding's inequality.

**Lemma 31.** With probability at least  $\left(1 - \exp\left(-\frac{(p^*)^2 M}{2}\right)\right)$ , it holds that

$$\sum_{i=1}^{M} \mathbb{1}\{\mathcal{J}(m) \neq \mathcal{J}(m-1)\} \ge \frac{1}{2}p^*M. \tag{9}$$

Next we prove Lemma 22.

**Proof of Lemma 22.** From [16, Chapter 4.10] we know that  $\{\mathcal{J}(t)\}_{t=0}^{\infty}$  has a corresponding jump process (also referred to as the embedded chain) that describes, conditioning on state changes, how  $\mathcal{J}(t)$  jumps among different states. We also know that this jump process is a simple random walk. Concretely, if  $\mathcal{J}(t-1) \neq \mathcal{J}(t)$ , we say a jump occurs at t. Let  $T_r$  denote the number of rounds that elapses between the r-th and r+1-th jumps of  $\mathcal{J}$ . Let  $\tilde{\mathcal{J}}(r)$  denote the state after r jumps. By definition,  $\left\{\tilde{\mathcal{J}}(r)\right\}_{r=0}^{\infty}$  is the jump process of  $\{\mathcal{J}(t)\}_{t=0}^{\infty}$ . It is easy to see that  $T_r$  is a geometric random variable with parameter  $p^*$ . Also,  $T_1, T_2, \cdots$  are i.i.d. distributed. By [16, Proposition 4.9], we know

$$\tilde{\mathcal{J}}(r) = \begin{cases} 0, & \text{if } r = 0; \\ \tilde{\mathcal{J}}(r-1) + 1, & \text{with probability } p_{+1}/p^*; \\ \tilde{\mathcal{J}}(r-1) - 1, & \text{with probability } p_{-1}/p^*. \end{cases}$$

Let  $\delta_r = \tilde{\mathcal{J}}(r) - \tilde{\mathcal{J}}(r-1)$  for any  $r \geq 1$ . It is easy to see that  $\delta_r$  is a Bernoulli random variable supported on  $\{-1,+1\}$  and  $\mathbb{P}\{\delta_r = 1\} = p_{+1}/p^*$ . For a given  $K \geq 0$  jumps, by Hoeffding's inequality, we know that with probability at least  $\left(1 - \exp\left(-\frac{K(p_{+1}/p^* - p_{-1}/p^*)^2}{8}\right)\right)$ , the following is true

$$\sum_{r=1}^{K} \delta_r \ge \frac{(p_{+1}/p^* - p_{-1}/p^*)}{2} K.$$

Setting  $K = \frac{1}{2}p^*M$ , we have

$$\sum_{r=1}^{K} \delta_r = \sum_{r=1}^{\frac{1}{2}p^*M} \left( \tilde{\mathcal{J}}(r) - \tilde{\mathcal{J}}(r-1) \right) = \tilde{\mathcal{J}}(\frac{1}{2}p^*M) - \tilde{\mathcal{J}}(0) = \tilde{\mathcal{J}}\left(\frac{1}{2}p^*M\right).$$

In addition, from Lemma 31, we know that with probability at least  $(1 - \exp(-((p^*)^2 M)/2))$ , it holds that

$$\sum_{i=1}^{M} \mathbb{1}\{\mathcal{J}(m) \neq \mathcal{J}(m-1)\} \ge \frac{1}{2}p^*M.$$
 (10)

Thus,  $\mathcal{J}(M) \geq \tilde{\mathcal{J}}\left(\frac{1}{2}p^*M\right)$ , proving the lemma.

**Lemma 32.** For any t > 1 round such that  $\mathcal{N}(t) = 0$  and at least one block is mined, let NB(t) be the number of blocks mined by the honest nodes in round t and AB(t) be the number of blocks mined by the corrupt nodes in round t. For  $\tilde{t} \triangleq \min\{t' : t' \leq t-1\}$  such that at least one block is mined, let  $AB(\tilde{t})$  be the number of blocks mined by corrupt nodes in round  $\tilde{t}$ . Then it is true that the number of longest chains at the end of round t is at most  $(NB(t) + AB(\tilde{t}))$ . In particular, if  $\mathcal{N}(t-1) = 0$ , then the number of longest chains at the end of round t is at most (NB(t) + AB(t)). Moreover, all of these longest chains end with blocks generated in round t.

*Proof.* One of the following two cases hold.

Case 1: Suppose that  $\mathcal{N}(t-1)=0$ . By Lemma 19, at the beginning of round t, the longest chains received (including their own local chains) by each of the honest node are of the same length. Thus, the number of longest chains at the end of round t is (NB(t) + AB(t)) – the number of blocks generated during round t. Case 2: Suppose that  $\mathcal{N}(t-1) \neq 0$ . Recall that  $\tilde{t} \triangleq \max\{t': t' \leq t-1\}$  such that at least one block is generated during round  $\tilde{t}$ . By Definition 17 and the fact that  $\mathcal{N}(t)=0$ , we know that  $\mathcal{N}(\tilde{t})=1$ . From Lemma 19, we know that, at the end of round  $\tilde{t}$ , the length of the adversarial longest chains of round  $\tilde{t}$  is at most one block longer than the local chains at the honest nodes. The adversary can choose either to hide these longest adversarial chains to some/all honest parities or to release those chains to all of the honest nodes. The number of such adversarial longest chains is at most  $AB(\tilde{t})$  – the number of blocks mined by the corrupt nodes in round  $\tilde{t}$ . Hence the number of longest chains at the beginning of round t is at most  $AN(\tilde{t}) + AB(t) + NB(t)$ .

**Lemma 33.** Let  $i_1$  and  $i_2$ , where  $i_1 \neq i_2$ , be two arbitrary honest nodes. For any  $(t-1) \geq 1$  such that at least one block is mined during round t-1, let  $C_1(t)$  and  $C_2(t)$  be the sets of longest chains received by honest nodes  $i_1$  and  $i_2$ , respectively, at round t before mining (including the chains sent by others at the end of round t-1 and forwarded at the beginning of round t). If  $\mathcal{N}(t-1)=0$ , it holds that

$$|\mathcal{C}_1(t) \cup \mathcal{C}_2(t)| \ge NB(t-1),\tag{11}$$

and

$$\max\{|\mathcal{C}_1(t)|, |\mathcal{C}_2(t)|\} \le NB(t-1) + AB(t-1) + AB(\widetilde{t-1}), \tag{12}$$

where  $\widetilde{t-1}$ : max $\{t': t' \leq t-2\}$ . Moreover, if  $\mathcal{N}(t-2) = 0$ , then

$$\max\{|\mathcal{C}_1(t)|, |\mathcal{C}_2(t)|\} \le NB(t-1) + AB(t-1). \tag{13}$$

Proof. Since  $\mathcal{N}(t-1) = 0$ , both  $\mathcal{C}_1(t)$  and  $\mathcal{C}_2(t)$  are subsets of the longest chains at the end of round (t-1). In addition, from Lemma 32, we know that the total number of longest chains at the end of round t-1 in the system is at most  $\mathrm{NB}(t-1) + \mathrm{AB}(t-1) + \mathrm{AB}(t-1)$ , proving Eq.(12). As each of the honest nodes who successfully mines a block during round (t-1) multi-casts its local chain to others, both  $i_1$  and  $i_2$  will receive all of the longest chains that end with an honest block. That is,

$$|\mathcal{C}_1(t) \cup \mathcal{C}_2(t)| \ge NB(t-1),$$

proving Eq.(11). Since  $\mathcal{N}(t-1)=0$ , it is easy to see that  $NB(t-1)\geq 1$ .

When  $\mathcal{N}(t-2) = 0$ , it holds that each of the honest node receives at least one longest chain up to the beginning of round t-1. Thus, the total number longest chains (including both the ones extended by the honest nodes and the ones extended by the corrupt nodes) is AB(t-1) + NB(t-1). Since a corrupt node can choose to not to multi-cast its local chain, it holds that

$$\max\{|C_1(t)|, |C_2(t)|\} \le NB(t-1) + AB(t-1).$$

Note that as a corrupt node can arbitrarily choose, independently of others, which subset of honest nodes to send its local chain, the honest nodes  $i_1$  and  $i_2$  can be two different subsets of the longest chains at the end of round t-1.

## F VDF-Scheme Proofs

The above VDF-based scheme presented in Section 5 gives us the following properties which will be crucial in obtaining our bounds for the case when adversaries are present. We present the proofs omitted from Section 5 in this section.

**Lemma 34.** For every round  $t \ge 0$ , the local chain of every honest node can contain at most one block per VDF output, with all but negligible probability in  $\lambda$ .

*Proof.* Item 5a ensures that the VDF outputs stored in the blocks are computed in strictly increasing order. This means that no chain verified by an honest node can contain two blocks which contain the same VDF output. Finally, with all but negligible probability in  $\lambda$  (by Definition 25), no two VDF outputs will be equal. The lemma follows.

**Lemma 35.** The adversary can add at most one block to the same chain (that will be verified by honest nodes) during any round  $r \ge 0$ , with all but negligible probability in  $\lambda$ . In other words, let t be the current round, any adversarial chain verified by honest nodes cannot have length greater than t.

*Proof.* By Lemma 34, no adversary can add two blocks with the same VDF output to the same chain. Thus, in any adversarial chain verified by honest nodes, the adversary can add at most 1 block per round, and the chain will have length at most t (where t is the current round).

**Lemma 36.** Let t be the current round. No node can add a block containing a VDF output,  $o_{t'}$ , from a round  $t' \geq t$ .

*Proof.* We prove this via induction. In the base case when the chain only contains the genesis block, the adversary has access only to the genesis block and no other blocks. In this case, the adversary could not have mined any other blocks because it did not have time to obtain the VDF output for the next block, with all but negligible probability in  $\lambda$ . Let our induction hypothesis be that the adversary cannot use  $d_t$  to mine a block in round t. We assume this is true for the t-th round and prove this true for the (t+1)-st round.

By our induction hypothesis, the adversary could not have computed any blocks for the t-th round using  $d_t$ . This means that the adversary obtained  $d_t$  at the beginning of the (t+1)-st round. Then, suppose the adversary computes a block for the (t+1)-st round. By Item 2 of Definition 27, the adversary could not have computed the output in time less than the duration of a round,<sup>8</sup> with all but negligible probability in  $\lambda$ . Thus, the adversary could not have mined any blocks using  $d_{t+1}$  until round t+2. Hence, no adversary can use  $d_{t+1}$  in mining any blocks during any round  $t'' \le t+1$ .

Proof of Theorem 26. Lemma 36 states that the adversary cannot mine a block with VDF output,  $o_{t'}$ , from a round t' > t where t is the current round. Thus, during round t, an adversary can only use VDF outputs,  $o_1, \ldots, o_{t-1}$ . Then, Lemma 34 and Lemma 35 ensure that no two blocks in a chain can contain the same VDF output. Then, the length of any chain accepted by an honest node has length at most t since it can contain at most one block using each of the VDF outputs,  $o_1, \ldots, o_{t-1}$  (plus the genesis block).

<sup>&</sup>lt;sup>8</sup>Recall we set the difficulty of the VDF to be the duration of a round.